Study on the Average Size of the Longest-Edge Propagation Path for Triangulations

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Abstract: For a triangle t in a triangulation τ, the "longest edge propagating path" Lepp(t), is a finite sequence of neighbor triangles with increasing longest edges. In this paper we study mathematical properties of the LEPP construct. We prove that the average LEPP size over triangulations of random points sets, is between 2 and 4 with standard deviation less than or equal to $\sqrt{6}$. Then by using analysis of variance and regression analysis we study the statistical behavior of the average LEPP size for triangulations of random point sets obtained with uniform, normal, normal bivariate and exponential distributions. We provide experimental results for verifying that the average LEPP size is in agreement with the analytically derived one.

1 INTRODUCTION

Triangulations are extensively used in a variety of applications such that finite element analysis, computer graphics, animation, visualization and computer aided design. Triangulation refinement for adaptive finite element methods is a process (algorithm) that for an input set S of target triangles with unacceptable finite element error, produces a valid refined triangulation (the triangles of S and some neighbor triangles are refined) such that the triangulation quality is maintained throughout the process. Refinement algorithms based on the longest edge bisection of triangles were developed for adaptive and multigrid finite element methods (Rivara, 1984a; Rivara, 1984b), which maintain the triangulation quality due to the mathematical properties of the longest edge bisection of triangles.

Later the LEPP construct was introduced and used in two directions: (1) to reformulate in a simpler and effective way previous longest edge refinement algorithms, which maintain the quality of the initial triangulation (Rivara, 1997; Bedregal and Rivara, 2014a); and (2) to develop LEPP Delaunay algorithms for the quality triangulation of planar straight line graph (PSLG) geometries, which improve a bad quality triangulation of the input PSLG data (Rivara, 1997; Bedregal and Rivara, 2014b; Rivara and Rodriguez-Moreno, 2019).

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The LEPP algorithms work as follows, given a target triangle t to be refined / improved, a finite sequence of increasing neighbor triangles (where t_{i+1} is neighbor of t_i by the longest edge of t_i, and the longest edge of t_{i+1} is greater than the longest edge of t_i) is computed. This sequence, called $Lepp(t)$, allows finding a local largest edge (terminal edge) in the mesh shared by a couple of terminal triangles (for an illustration see Figure 1). Then, over the couple of terminal triangles, a local refinement operation is performed, which locally improves the triangulation quality. The local refinement operations used are either the longest edge bisection of the terminal triangles, or the Delaunay insertion of a point selected in the interior of the terminal triangles (terminal edge midpoint or centroid of the terminal triangles).

It has been proven that the LEPP algorithms produce optimal size triangulations due to the improvement properties of the local operations performed inside the terminal triangles, which in turn implies that the average size (number of triangles) of Lepp(t) decreases and tends to be 3 as the refinement proceeds (Bedregal and Rivara, 2014a; Rivara and Rodriguez-Moreno, 2019).

We should emphasize that the LEPP construct resulted to be an effective and simple tool for mesh improvement both in 2D and 3D suitable to be included in mesh generation software and for parallelization. Discussions on these issues can be found in (Rivara and Rodriguez-Moreno, 2019; Balboa et al., 2019).

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In this paper we study the average LEPP size over static triangulations which are the input meshes required in different complex applications such those related with finite element methods. More specifically, we compute the worst LEPP size, and the average LEPP size for triangulations of sets of random points in 2D.

It is important to study the average LEPP size because this is the most frequent case in the triangulation refinement process. This is similar to what happens with the Quicksort algorithm, where on the average case the execution time is optimal $n \cdot \log(n)$ to sort n items and has better performance than its competitors, but it is bad in the worst case n^2 (Sedgewick and Wayne, 2011; Sedgewick and Flajolet, 2013).

2 RELATED WORK

Formally, Lepp(t) is defined as follows: for any triangle t_1 in τ, the longest edge propagating path (Lepp(t_1)) is defined as the finite sequence of increasing triangles $\{t_i\}_{i=1}^n$ such that t_{i+1} is the neighbor of t_i by its longest edge, and where the longest edge of t_{i+1} is greater than the longest edge of t_i. The LEPP path allows to find an associated local largest edge E (terminal edge) in τ, either shared by two terminal triangles t_{n-1}, t_n, or E being a boundary terminal edge (longest edge of t_n). For an illustration see Figure 1, where AB is an interior terminal edge.

LEPP algorithms (Rivara, 1997; Bedregal and Rivara, 2014a; Bedregal and Rivara, 2014b; Rivara and Calderon, 2010; Rivara and Rodriguez-Moreno, 2019) proceed as follows: for each target triangle t to be refined / improved, the longest edge propagating path (LEPP) is computed to find a couple of terminal triangles sharing a common longest edge AB (terminal edge) as shown in Figure 1. Then a point is selected inside the terminal triangles (terminal edge midpoint or terminal triangles centroid) and inserted in the mesh either by triangle longest edge bisection or by Delaunay insertion. The process is repeated until the target triangle is refined.

Note that over the terminal triangles local refinement operations are performed which locally improve the triangulation quality.

LEPP Bisection Algorihtm. Due to the improvement properties of the iterative longest edge bisection of triangles, refined triangulations that maintain the triangulation quality (bounded smallest angle) are obtained, while the proportion of quality triangles increases as the refinement proceeds. Based on these properties, it has been proven that optimal size triangulations are obtained (Bedregal and Rivara, 2014a).

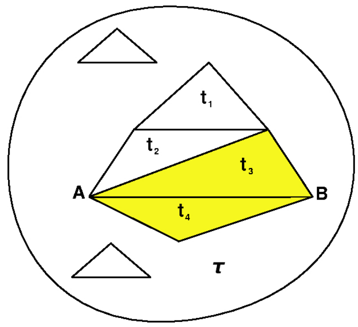


Figure 1: Lepp(t_1) = $\{t_1, t_2, t_3, t_4\}$ in triangulations τ allows to find a local largest edge (terminal edge) AB and a couple of terminal triangles (t_3, t_4) over which refinement / improvement operations are performed.

LEPP Delaunay Algorithms. These algorithms produce quality Delaunay triangulation, based on the Delaunay insertion of special points inside the terminal triangles (terminal edge midpoint or centroid of the terminal triangles). For the LEPP centroid algorithm, termination, and optimal size property were proven. Furthermore the size of the refined triangulation is almost equal independently of the triangle processing order (Rivara and Rodriguez-Moreno, 2019). In 3-dimensions a mesh improvement algorithm based on the extensions of some of these ideas to 3-dimensions have been also discussed (Balboa et al., 2019).

2.1 On the LEPP Size

In the LEPP algorithms, for each target triangle t (to be refined or improved) the LEPP computation is repeatedly performed until the triangle is destroyed. Thus the number of points inserted to refine / improve triangle t roughly depends on the LEPP size. For the LEPP algorithms it has been proven that the average LEPP size is small and tends to be 3 as the refinement algorithm proceeds. This was firstly proven for triangulations obtained by the LEPP bisection algorithm (Bedregal and Rivara, 2014a) and later extended to the LEPP Delaunay algorithms (Rivara and Rodriguez-Moreno, 2019).

In this paper we state results on the average LEPP size for triangulations. Assuming that the probability of finding a longest edge neighbor in the LEPP path is $p \geq \frac{1}{3}$, we prove that the average LEPP size is between 2 and 4. We also present a statistical study over triangulations of random point sets, showing that the average LEPP size is in agreement with the theory.

3 THEORETICAL STUDY ON THE AVERAGE LEPP SIZE

3.1 Worst Case on the LEPP Size

In the worst case, the LEPP size is equal to the size of the triangulation τ, the number of triangles n, which occurs for the special case where there exists a smallest triangle t_1, such that, $Lepp(t_1)$ covers τ and size of $Lepp(t_1)$ is equal to n. For an example see Figure 2.

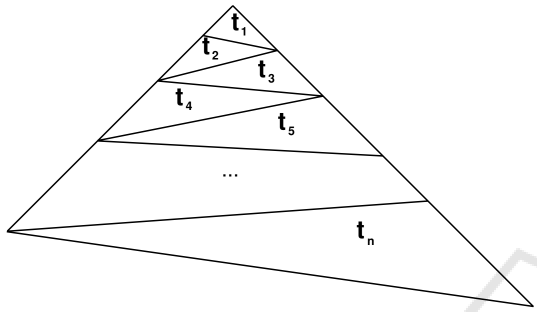


Figure 2: Worst case of the maximum LEPP size (with n triangles).

Here the average LEPP size is equal to

$$\mu_{LEPP} = \frac{\sum_{i=1}^{n} Lepp(t_i)}{n} = \frac{\sum_{i=1}^{n} \mathrm{i}}{n} = \frac{n+1}{2} \quad (1)$$

Note that in this case the average LEPP size increases with n.

3.2 Theoretical Average LEPP Size

Here we calculate the average LEPP size of triangulations. This extends and improves results presented in (Vilca, 2009; Vilca et al., 2010), the following more general theorems are formulated in terms of a parameter p. In section 4 we also include statistical discussion on empirical results.

Firstly we need to introduce the following definition:

Definition 1. *For any triangle t of longest edge E and neighbor triangle t^* with edges $E_1^* \geq E_2^* \geq E_3^*$, we will say that the neighbor triangle t^* by the edge E is of type A, B, C if the following conditions hold:*

- *t^* is of type A if $E = E_3^*$*
- *t^* is of type B if $E = E_2^*$*
- *t^* is of type C if $E = E_1^*$*

Theorem 1. *Let τ a triangulation constructed from a set of random points in 2D, if for any triangle in τ the probability of having a neighbor triangle of type C is p, then the average LEPP size is $\frac{p+1}{p}$ with standard deviation equal to $\frac{\sqrt{1-p}}{p}$, the skewness coefficient is equal to $\frac{2-p}{\sqrt{1-p}}$, and the kurtosis coefficient is equal to $\frac{p^2-6p+6}{1-p}$.*

Proof. If for any triangle T_i the probability of having a neighboring triangle T_{i+1} of type C is p, then, the probability of having a neighboring triangle of type A or B is q (where $p+q=1$).

It is easy to see that $Lepp(T_1)$ is an ordered sequence of triangles composed of three parts: (i) An initial triangle T_1 without type because the first triangle does not have a predecessor neighbor triangle. (ii) Followed indistinctly by zero or more triangles of type A or B, and (iii) ending with a triangle T_n of type C. Then the minimum length of the LEPP is two (composed of the first triangle without type and the last type C triangle).

The above sequence corresponds to the geometric distribution $P(n) = q^k p$, where the mean on the average number of triangles of type A or B is $\mu = \frac{q}{p}$, to which a value 2 should be added, according to items (i) and (iii) on the number of elements in the LEPP path. Thus, the average LEPP size is $\frac{p+1}{p}$. Furthermore the variance, third and fourth moments are given by:

$$V(G) = \frac{q}{p^2} \quad (2)$$

$$\mu_3 = \frac{q(2-p)}{p^3} \quad (3)$$

$$\mu_4 = \frac{q(p^2-9p+9)}{p^4} \quad (4)$$

Finally, from equation 2, the standard deviation of the average LEPP size is $\sigma = \sqrt{V(G)} = \frac{\sqrt{1-p}}{p}$, while the coefficients of skewness and kurtosis are:

$$Sk = \frac{\mu_3}{\sigma^3} = \frac{2-p}{\sqrt{1-p}} \quad (5)$$

$$Ku = \frac{\mu_4}{\sigma^4} - 3 = \frac{p^2-6p+6}{1-p} \quad (6)$$

□

Remark Theorem 1 does not consider triangles t_e with boundary longest edges ($Leep(t_e) = 1$). Therefore, this is indeed a result over an infinite mesh.

Theorem 2. *Let τ be a triangulation constructed from a set of random points in 2D and let μ_{LEPP} be the average LEPP size, then $2 \leq \mu_{LEPP} \leq 4$ with standard deviation between 0 and $\sqrt{6}$, assuming that for any triangle in τ the probability of having a neighbor triangle of type C is $p \geq \frac{1}{3}$.*

Proof. In the theorem 1 the average LEPP size is $\frac{p+1}{p} = 1 + \frac{1}{p}$, furthermore, by definition of probability (p is from 0 to 1) and the condition $p \geq \frac{1}{3}$ in τ: $1/3 \leq p \leq 1$, therefore, $2 \leq 1 + \frac{1}{p} \leq 4$ and $0 \leq \frac{\sqrt{1-p}}{p} \leq \sqrt{6}$. □

4 EXPERIMENTAL RESULTS ON THE AVERAGE LEPP SIZE OVER TRIANGULATIONS OF SETS OF RANDOM POINTS

To perform the analysis of variance (ANOVA) and linear regression, the following three conditions must be checked (Diez et al., 2015) on the data:

- Test of independence. The observations are independent within and across groups.
- Normality test. The data within each group are nearly normal.
- Test for homogeneity of variance. The variability across the groups is about equal.

Without the tests, the analysis of variance and linear regression are not valid. For the hypothesis tests, the practical significance level of 0.05 (even 0.01) were used, in accordance with our criteria and considering the consequences associated with Type 1 and Type 2 errors see for more details (Diez et al., 2015).

Analogous results and conclusions were obtained for all the distributions. For illustrative purposes, only the result of the uniform distribution are shown.

4.1 Generation of Random Points for Computing the Average LEPP Size

In order to obtain LEPP results (mainly on propagation length), a C++ program was implemented to generate random points in a two-dimensional space, by using the most known probability distributions (Thomopoulos, 2018) see Figure 3:

- Uniform distribution on the square.
- Normal or Gaussian distribution.
- Bivariate normal distribution.
- Exponential distribution.

Then, by using the CGAL library (The Computational Geometry Algorithms Library https://www.cgal.org), triangulations were built for each distribution and for each case size (according to the number of points considered). The following libraries were used: "2D Triangulation" (Yvinec, 2019) and "2D Triangulation Data Structure" (Pion and Yvinec, 2019). We implemented a program to calculate the means of LEPP and other statistical measures.

4.2 Test of Independence

Independence within groups. The average LEPP size of each experiment is independent, and the generation of vertices is random and independent.

Independence between groups. This assumption is fulfilled, because the random generation of vertices, and because the average LEPP sizes between groups are independent.

4.2.1 Runs Test for Randomness and Kolmogorov-Smirnov Goodness of Fit

The Kolmogorov-Smirnov test is used to decide if a sample comes from a population with a specific distribution, therefore, it was required for out different distributions. The following hypothesis were used to perform the randomness test (runs test) on each set of generated vertices (for each distribution):

H_0: the sequences of vertices for the empirical uniform, normal, bivariate normal, and exponential distributions were generated randomly.

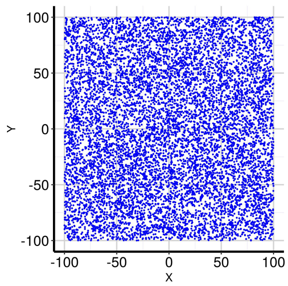
H_1: the sequences of vertices for the empirical uniform, normal, bivariate normal, and exponential distributions were not generated randomly.

Table 1: Runs and goodness of fit tests for the vertices of the uniform distribution.

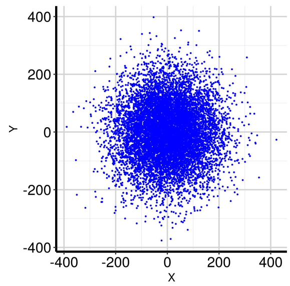
No.	Runs test		Goodness of fit test	
	p-value x	p.value y	p-value x	p-value y
1	0.6030	1.0000	0.2552	0.144
2	0.9681	0.6745	0.9482	0.952
3	0.6030	0.9522	0.9746	0.8195
…				
40	1.0000	0.2627	0.8497	0.2315

The results are presented in Table 1. Note that in the column "runs test", the p-values are greater than the significance level of 0.05 (even for 0.01). Consequently the decision is not to reject the null hypothesis. It is concluded that the data are random at the significance level of 0.05.

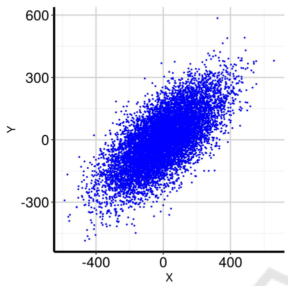
In the column "goodness of fit test" the Kolmogorov-Smirnov results are presented. It is observed that the p-values are greater than the significance level of 0.05, thus the decision is not to reject the null hypothesis, that is, there is not enough evidence to conclude that the set of vertices do not follow



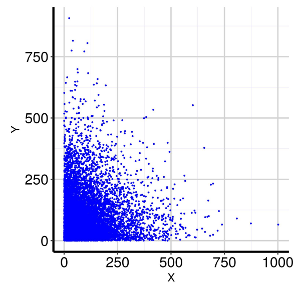
(a) Uniform distribution on a square.



(c) Normal distribution.



(d) Bivariate normal distribution.



(e) Exponential distribution.

Figure 3: Scatter plots generated using distribution functions.

the uniform distribution. Results of the uniform distribution are shown in Table 1. The same conclusions were obtained on all the distributions.

4.3 Test for Homogeneity of Variance

We used the Levene test which is an inferential statistics test used to evaluate the equality of the variances for a variable calculated for two or more groups (the most common assessment).

Table 2: Result of the Levene test for groups formed with the uniform distribution, test used to evaluate the equality of the variances needed for an ANOVA.

	Df	F value	Pr(>F)
group	19	10.58	0.0000
	780		

See table 2 where "Df", "F value" and "Pr(>F)" respectively correspond to the degrees of freedom, the value of the test statistic, and the p-value for the test. The p-values are less than the significance level of 0.05 in all the distributions, which indicates that the assumption on equality of variances between groups is not met.

For each case size (for example for 10000 points), the experiment was repeated 40 times, therefore, groups are of equal size (40). We can ignore the homogeneity of variance assumption if we have equal sample sizes for each group.

4.4 Normality Test

The Shapiro-Wilk test was used for testing normality in each sample/group. This provides better power than the Kolmogorov-Smirnov test even after the Lilliefors correction (Steinskog et al., 2007).

Table 3: Result of the Shapiro-Wilk test for the groups formed with the uniform distribution.

	Grupo	W	P.value
1	G010	0.97	0.39
2	G020	0.97	0.40
3	G030	0.96	0.18
...			
20	G200	0.98	0.55

There is no enough evidence to conclude that the assumption of normality in all the distributions is not met, because the p-values are greater than the significance level of 0.05. Results of the uniform distribution are shown in Table 3.

4.5 Empirical Study on the Average LEPP Size

In this section, we analyze the average LEPP size obtained for the triangulation of each set of vertices, that is, for randomly generated points with different distribution functions. Groups of vertices of different sizes were formed, starting with groups of size 10000 and using successive increments of 10000 up to obtain groups of 200000 points. For each group size (case size) the experiment was repeated 40 times in order to obtain general results avoiding non-compliance with the assumptions of the analysis of variance. Then, summaries of the means of each group of triangulations, the maximum LEPP, the average LEPP, were computed.

Table 4: Means obtained for each group of triangulations formed with the uniform distribution (number of vertices in thousands).

No. vertices	mTriangles	mMax.L	mLEPP	mSD.L
10	19974.05	11	3.0485	0.0115
20	39971.78	12	3.0512	0.0099
30	59971.50	12	3.0561	0.0075
40	79970.52	12	3.0587	0.0059
50	99969.60	12	3.0590	0.0063
60	119968.73	12	3.0597	0.0057
70	139969.30	13	3.0607	0.0044
80	159967.25	13	3.0619	0.0049
90	179968.05	13	3.0626	0.0043
100	199967.52	13	3.0629	0.0043
110	219967.62	13	3.0637	0.0046
120	239966.98	13	3.0645	0.0037
130	259967.25	13	3.0641	0.0033
140	279967.03	13	3.0648	0.0034
150	299967.65	14	3.0650	0.0034
160	319966.22	13	3.0652	0.0029
170	339965.25	14	3.0662	0.0033
180	359965.83	14	3.0662	0.0030
190	379966.58	14	3.0664	0.0037
200	399965.50	14	3.0661	0.0028

Table 4 presents results of the uniform distribution. This is composed of 20 rows, each of them including the summary of the obtained averages for each group of 40 experiments, which were formed from a fixed number of vertices in each group. The column No.vertices indicates the number of vertices used in each experiment group, the column mTriangles shows the average number of triangles obtained for the group, while, mMax.L is the average of the maximum LEPP of the group's experiment, mLEPP is the average LEPP size, and mSD.L is the average of the standard deviations for each group. Note that the average LEPP size remains almost constant, in relation to the maximum LEPP size, which increases slightly because each row represents a group of greater number of vertices.

Figure 4 and 5 show the variability of the average LEPP size. Note that the variability is greater for groups formed with less vertices and is reduced when the number of vertices increases. The same behavior was obtained for all the distributions.

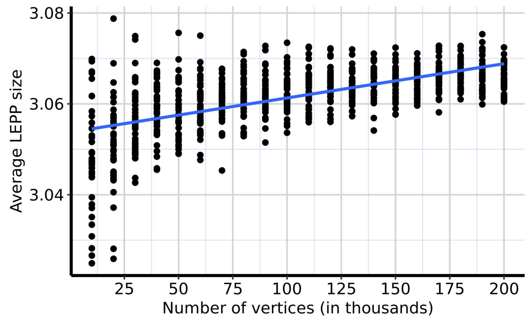


Figure 4: Scatterplot of the average LEPP in relation with the number of vertices formed with the uniform distribution.

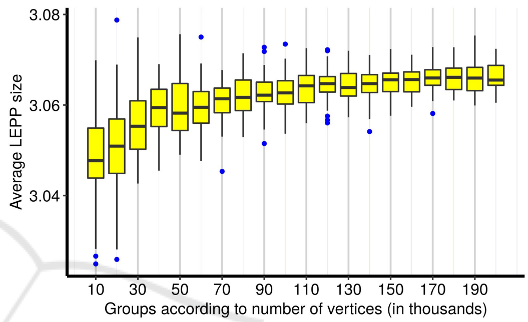


Figure 5: Box plot of the average LEPP size according to the number of vertices (groups) - formed with the uniform distribution.

4.6 Difference of Means Test

The following hypothesis was used to perform the equality test.

H_0: all the LEPP means of the groups formed with the (uniform, normal, normal bivariate, or exponential) distributions are equal.

H_1: at least one pair of the LEPP means formed with the (uniform, normal, normal bivariate, or exponential) distributions are different.

Table 5: Summary result of the analysis of variance for the difference of means between the groups formed with the uniform distribution.

	Df	F value	Pr(>F)
Group	19	33.34	0.0000
Residuals	780		

In Table 5 we present results on the ANOVA analysis, Note that for data groups of the uniform distribution, the p-value is less than the significance level of 0.05, Therefore, the null hypothesis is rejected even for the significance level of 0.001 and it is concluded that there is significant difference between the groups. The same conclusion was obtained for all the distributions.

4.7 One-sample T-test for the Groups of Vertices Generated According to Each Probability Distribution

The following hypothesis was used to perform the t test on each set of vertices.

H_0: The average LEPP size of the triangulations constructed from the sets of points generated with probabilistic distributions is equal to four.

H_1: The average LEPP size of the triangulations constructed from the sets of points generated with probabilistic distributions is less than four.

Table 6: Test t for the vertices of each group and each distribution (number of vertices in thousands).

No. vertices	Uniform p-value	Normal p-value	N.Bivariate p-value	Exponential p-value
10	6.08e-77	9.50e-74	2.53e-77	2.85e-73
20	2.27e-79	2.23e-80	5.38e-85	1.34e-79
30	5.88e-84	9.25e-88	1.05e-82	1.96e-85
...				
200	2.65e-100	6.73e-100	2.64e-96	1.00e-94

In Table 6, where the columns represent each distribution and the rows represent the different groups, one can see that the p-values are less than the significance level of 0.05, and the decision is to reject the null hypothesis. We conclude that the average LEPP size is significantly less than 4. In the same way, tests were conducted and it was concluded that the average LEPP size is significantly greater than 2. Therefore, the results of the Theorem 2 are empirically and statistically confirmed (at the significance level of 0.05).

4.8 Regression Analysis

Linear models can be used for prediction or to evaluate whether there is a linear relationship between two numerical variables (Diez et al., 2015). We have used linear, quadratic, logistic and logarithmic regressions to study the data.

Scatterplot formed with the uniform distribution and regressions computed on the average LEPP size with respect to the number vertices, are shown in Figure 6 ($n \leq 200000$). The logistic curve slightly overlaps the linear one, and Figure 7 clarifies this behavior (zoom out).

Note that there is a significant relationship between the average LEPP size and the number of vertices x, in the four regression models, because the p-values are much less than 0.05 (significance level) as shown in Table 7.

The Figures 6 shows a very weak upward trend in the data, so slight we can hardly notice it. In the linear regression model, the average LEPP size tends to grow rapidly and without limit, so it is discarded (see Figure 7). The quadratic regression attains a maximum and then decreases producing negative values when $n \to \infty$, so it is discarded. The logistic regression model fits well the data and has asymptotic limit equal to 4 (LEPP=4 as the theory predicts). In exchange, the logarithmic regression model has the best adjusted R squared, that is, it fits the data better in relation to the other models, and keeps well the LEPP average, as shown in Figures 6 and 7.

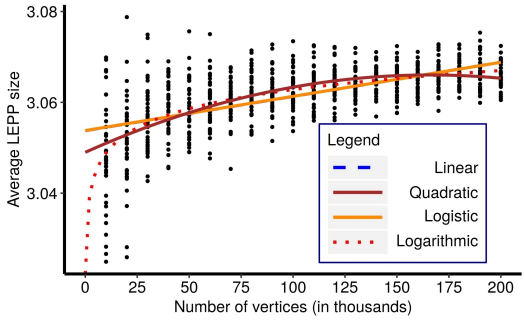


Figure 6: Scatterplot formed with the uniform distribution and regression models on the average LEPP size with the number vertices (the logistic curve slightly overlaps the linear one).

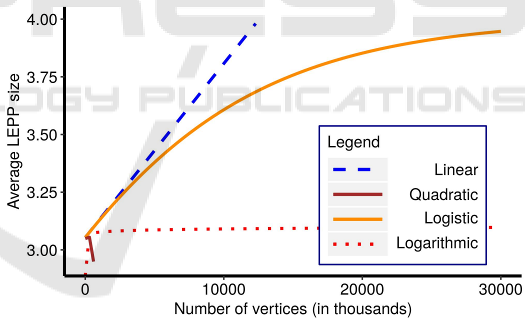


Figure 7: Projections of the regression models of the uniform distribution (zoom out). The quadratic regression decreases and produces negative values when $n \to \infty$.

Table 7: Summary of regression model results. First row is linear regression, second row is quadratic regression, third row is logistic regression, fourth row is logarithmic regression.

Regression model for LEPP	Adj.R^2	p-value
$3.054 + 7.535e5x$	0.36	1.99e-79
$3.049 + 0.000205x - 6.19e\text{-}7x^2$	0.42	2.74e-96
$\frac{4}{1+e^{-1.172-0.000104x}}$	0.36	1.57e-79
$3.035 + 0.0042\log_2 x$	0.44	1.42e-102

5 CONCLUSIONS

Under an assumption on equal probability for neighbor triangles, we have proven that the average LEPP size over triangulations of random points sets, is between 2 and 4 with standard deviation between 0 and $\sqrt{6}$. We also presented a extensive statistical study over triangulation of random point sets generated with four distribution functions (uniform, normal, bivariate normal and exponential), showing that in practice, the average LEPP size is in agreement with the theory. Since in computational terms the LEPP cost is constant $\Theta(1)$, these results contribute to support LEPP algorithms and LEPP techniques for triangulation improvement in 2-dimensions. More research is needed to study the distribution of terminal edges in the mesh.

As future research we also suggest to study the average LEPP size in 3-dimensions, which seems to behave analogously to 2-dimensions in practice. This is a more difficult problem since in 3-dimensions the improvement properties of the longest edge bisection of tetrahedra have not been yet stated. (Rivara and Levin, 1992; Rivara and Palma, 1997).

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