Co-Op Advertising with Two Competing Retailers: A Feedback Stackelberg-Nash Game

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Abstract: A nearly explicit feedback Stackelberg-Nash equilibrium is obtained in a dynamic distribution channel consisting of a manufacturer and two competing asymmetric retailers engaged in promoting the manufacturer's product to be sold through the retailers. The manufacturer decides on its support for the retailers' advertising activities by announcing cooperative advertising subsidies called "participation rates." The retailers compete for market share by selecting advertising efforts. We formulate the problem as a Stackelberg-Nash differential game and reduce it to merely solving a set of algebraic equations. We find that the manufacturer should offer the cooperative advertising policy to only one retailer and even then, only when a "participation threshold" depending on the model parameters is exceeded. We identify the levers that determine the optimal participation rate. Furthermore, we obtain important insights into how sensitive the optimal solution is with respect to the parameters. Moreover, we show that retail-level competition induces the manufacturer to offer a higher level of support to the supported retailer and over a wider range of parameters when compared to the results obtained in a one-manufacturer, one-retailer setting studied in the literature.

1 INTRODUCTION

Manufacturer-retailer relationships concern about the decisions of each channel member affect the other's profitability and strategy choices. For example, the retailer advertises the manufacturer's product to increase its sales, but it may not do so to the extent that the manufacturer might prefer. Then, the manufacturer might provide incentives to the retailer. The situation is complicated even further if there is more than one retailer with common customer base carrying the manufacturer's product. This case is not uncommon in situations where territories are not exclusive. The manufacturer and the retailers make an effort to maximize their individual profits.

Manufacturers often use cooperative advertising to influence their retailers' advertising decisions. Cooperative advertising is an arrangement whereby a manufacturer agrees to reimburse a portion of the advertising expenditures incurred by retailers for selling its product (Bergen and John 1997).

Cooperative advertising programs can be a significant part of the advertising budgets of manufacturers. By some estimates, more than $25 billion was spent on cooperative advertising in 2007, compared to $15 billion in 2000 and $900 million in 1970 (Nagler 2006), and approximately 25-40% of all manufacturers used this arrangement (Dant and Berger 1996). The importance of understanding cooperative advertising programs in manufacturer-retailer relationships is based on these figures.

Dutta et al. (1995) conduct an empirical analysis of cooperative advertising plans offered by manufacturers to their retailers and report that the average participation rate over all product categories is 74.6%. More importantly, they find that the participation rate differs from industry to industry—it is 88.38% for consumer convenience products, 69.85% for consumer nonconvenience products, and 69.02% for industrial products. In this paper, we make a theoretical contribution to the cooperative advertising literature by endogenously determining the optimal participation rate in the face of competition as a function of various firm- and industry-level parameters and analyzing how these parameters affect a manufacturer's participation rate policy.

We model a dynamic distribution channel in which a manufacturer sells a product through one or

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both of two independent and competing retailers. The retailers choose their advertising efforts after the manufacturer decides the extent of its support for each retailer's advertising activity. This is called "participation rate," i.e., the portion of the retailer's advertising expenditures that the manufacturer will subsidize. The manufacturer first chooses its participation rates. Taking the participation rates into consideration, the two retailers simultaneously choose their local advertising levels. Sales are then realized based on the advertising efforts.

As our dynamics, we use a competitive extension of the Sethi (1983) advertising model along the lines suggested by Sorger (1989) and Prasad and Sethi (2004). The leader-follower sequence in the channel is formulated as a Stackelberg differential game. Stackelberg differential games are quite difficult to solve, even without the competition among the followers involving a Nash differential game amongst them. Often, only an open-loop solution is obtained, which is, in general, time inconsistent; e.g., see Dockner et al. (2000).

We provide a nearly explicit (time-consistent) feedback Stackelberg equilibrium for a cooperative advertising problem with retail competition. Furthermore, the equilibrium is reduced to merely solving a set of algebraic equations. We identify the levers that determine the optimal participation rate and obtain important insights into how sensitive the optimal solution is with respect to the parameters. Importantly, it makes possible to study the effect of retail-level competition on the behavior of the manufacturer. The proofs of the results in the paper are relegated to Appendix.

2 RELATED LITERATURE

In the context of cooperative advertising, we first review static models in the literature; e.g., see Berger (1972), Berger and Magliozzi (1992), Dant and Berger (1996), Bergen and John (1997), Kali (1998), Kim and Staelin (1999), Huang et al. (2002). In an early paper, Berger (1972) models cooperative advertising as a wholesale-price discount and make out that the manufacturer can use cooperative advertising to make higher profits. Dant and Berger (1996) extend the Berger (1972) model to study the role of cooperative advertising in franchising systems. They obtain that the franchisor and the franchisee would be better off if both of them jointly determine their cooperative advertising contributions than if they were to maximize their profits separately. Bergen and John (1997) explore the impact of advertising spillover and manufacturer and retailer differentiation on the participation rate. They show that the participation rate should be higher for less differentiated retailers, more differentiated brands, and more upscale products in a product category. For understanding how brand name advertising, local advertising, and the participation rate effect in cooperative advertising, Huang et al. (2002) analyze two models—a traditional model with the manufacturer as the Stackelberg leader and another where the manufacturer and the retailer form a cooperative advertising partnership. These results show that the total channel profits and the investments in national and local advertising are higher in the partnership setting than in the traditional case. There are many dynamic models of cooperative advertising in the literature (e.g., Jørgensen et al. 2000, 2001, 2003, Jørgensen and Zaccour 2003, Karray and Zaccour 2005, He et al. 2007, He et al. 2009). Jørgensen et al. (2000) consider a setting in which a manufacturer and an exclusive retailer decide on advertising that has both long-term and short-term effects on sales. They find that both channel members attain higher profits when the manufacturer supports both types of retail advertising than if it were to provide only partial advertising support. Jørgensen and Zaccour (2003) analyze a goodwill model of advertising in which there is no natural channel leader. Using a dynamic incentives approach, they show that the use of cooperative advertising can generate a Pareto-optimal joint profit maximization outcome. Jørgensen et al. (2003) analyze a model with a manufacturer, who invests in national advertising to promote the brand's image, and a retailer, who invests in local advertising that damages the brand's image. They show that it is optimal for the manufacturer to use cooperative advertising if the brand's image is sufficiently low, or if the harm to the brand's image from the retailer's advertising efforts is low. Karray and Zaccour (2005) model advertising at both the manufacturer and retailer levels. They consider a retailer that sells two products—the manufacturer's product and a private label at a lower price—and show that the manufacturer can use cooperative advertising to mitigate the negative impact of the retailer's private label. The papers most related to our work are He et al. (2011,2012) and Chutani and Sethi (2012). We will specify how these are related to the paper in the next section where we develop our model.

3 MODEL

We consider a distribution channel with a single manufacturer who sells a product to one or both of two independent and competing retailers, Retailer 1 and Retailer 2. Let $x(t)$ denote the market share of Retailer 1 at time $t \geq 0$, which depends on its own and its competitor's advertising efforts. Accordingly, the market share of Retailer 2 at time t is $1-x(t)$. The manufacturer supports the retailers' advertising activities by sharing a portion of the retailers' advertising expenditures. This support, termed the participation rate, for Retailer i is denoted θ_i.

In Table 1, we list the notation above and the remaining notation used in this paper.

Table 1: Notation.

t	Time t, $t \geq 0$
$x(t) \in [0,1]$	Retailer 1's proportion of the market at time t
x_0	Initial market share of Retailer 1
$u_i(t) \geq 0$	Retailer i's advertising effort rate at time t, $i = 1,2$
$\theta_i(t) \geq 0$	Manufacturer's participation rate for Retailer i at time t
$\rho_i > 0$	Advertising effectiveness parameter
$\delta_i \geq 0$	Market share decay parameter
$r > 0$	Discount rate of the two retailers and the manufacturer
$m_i > 0$	Gross margin of Retailer i
$M_i \geq 0$	Gross margin of the manufacturer from Retailer i
V_i, V	Value functions of Retailer i and the manufacturer, respectively
$f'(x)$ or f'	df/dx for a differentiable function $f(x)$
$f(x)^+$ or f^+	$\max\{f(x), 0\}$ for a function $f(x)$

We will assume $M_1 \geq M_2 \geq 0$ without loss of generality. The case $M_1 = M_2 \geq 0$ reduces our model to that of Prasad and Sethi (2004). Thus, we contribute by studying the case $M_1 > M_2 \geq 0$, without loss of generality, since the case $M_2 > M_1 \geq 0$ can be treated simply by re-labeling the firms.

The special case $M_1 > M_2 = 0$ arises when the manufacturer sells only through Retailer 1, and Retailer 2 represents an independent competing channel, and it has been studied in He et al. (2011). Since their aim was to examine simply the effect of competition from an independent retailer, they did so under the simplifying assumption that $\rho_1 = \rho_2$, $m_1 = m_2$, and $\delta_1 = \delta_2$. We relax these assumptions in this paper.

Another related paper is He et al. (2012) that assumes $M_1 = M_2$, whereas we generalize their model by relaxing this requirement. We should however mention that they also study the case of three retailers each giving the same margin to the manufacturer.

Finally, Chutani and Sethi (2012) have studied a retail market oligopoly with N retailers. However, their sales-advertising dynamics is represented by an extension of the 1983 Sethi model by Erickson (2009).

The advertising expenditure is quadratic in the advertising effort $u_i(t)$, $i = 1,2$, and the manufacturer's and Retailer i's advertising expenditure rates at time t are given by $\theta_1(t)u_1^2(t) + \theta_2(t)u_2^2(t)$ and $(1-\theta_i(t))\,u_i^2(t)$, $i = 1,2$, respectively. The assumption of a quadratic cost function is common in the literature and implies diminishing marginal returns to advertising expenditure; e.g., see Deal (1979), Sorger (1989), Chintagunta and Jain (1992), Prasad and Sethi (2004), Bass et al. (2005), He et al. (2009).

The market share dynamics of Retailer 1 is given by

$$\begin{aligned} \dot{x}(t) = \quad & \rho_1 u_1(t)\sqrt{1-x(t)} - \rho_2 u_2(t)\sqrt{x(t)} - \delta_1 x(t) \\ & + \delta_2(1-x(t)), \quad x(0) = x_0 \in [0,1], \end{aligned} \tag{1}$$

where the advertising response constant ρ_i determines the effectiveness of Retailer i's advertising activity, $i = 1,2$, and the market share decay constants δ_1 and δ_2 determine the rate at which its consumers are lost and gained, respectively, due to churn. This specification, characterized by the square-root feature introduced in the Sethi (1983) model, has the same desirable properties of concave response with saturation as the Vidale-Wolfe (1957) model. Note that the market share is non-decreasing in the retailer's own advertising effort and non-increasing in the competitor's advertising effort. Moreover, the specified concave response has been validated in empirical studies by Chintagunta and Jain (1992), Naik et al. (2008), and Erickson (2009).

The present-valued profits of Retailer 1, Retailer 2 and the manufacturer can be expressed, respectively, as

$$\pi_1 = \int_0^\infty e^{-rt}\left(m_1 x(t) - (1-\theta_1(t))\,u_1^2(t)\right)dt, \tag{2}$$

$$\pi_2 = \int_0^\infty e^{-rt}\left(m_2(1-x(t)) - (1-\theta_2(t))\,u_2^2(t)\right)dt, \tag{3}$$

and

$$\begin{aligned} \pi = \max_{\theta_1(t),\, \theta_2(t),\, t\geq 0} \int_0^\infty e^{-rt} & [(M_1 x(t) + M_2(1-x(t)) \\ & -\theta_1(t)(u_1(t))^2 - \theta_2(t)(u_2(t))^2]dt \end{aligned} \tag{4}$$

subject to the state equation (1).

In a feedback Stackelberg-Nash equilibrium for our infinte-horizon problem, the manufacturer determines his strategy in the feedback form as $\theta_i(x) : [0,1] \to [0,1], i = 1,2$, and the retailer i's strategy is based on the state as well as the manufacturer's decision, and is therefore of the form $u_i(x, \theta_1(x), \theta_2(x)) : [0,1]^3 \to [0,\infty]$, $i = 1,2$.

Let Θ_i and $U_i,\ i=1,2$, denote the spaces of such strategies of the manufacturer and the retailers, respectively. Then given $\theta_i \in \Theta_i$ and $u_i \in U_i,\ i=1,2$, we denote by $x^{t,y}(s;\theta_1,\theta_2,u_1,u_2),\ s \geq t$, the solution of the equation

$$\dot{x}(s) = \rho_1 u_1(x(s))\sqrt{1-x(s)} - \rho_2 u_2(x(s))\sqrt{x(s)}, \quad x(t)=y. \qquad (5)$$

Let

$$\Pi_i^{t,y}\Big[\theta_1(\cdot),\theta_2(\cdot),u_1(\cdot,\theta_1(\cdot),\theta_2(\cdot)),u_2(\cdot,\theta_1(\cdot),\theta_2(\cdot))\Big] = \int_0^\infty \exp^{-r(s-t)}\Big[m_i x^{t,y}(s;\theta_1,\theta_2,u_1,u_2) - (1-\theta_i(x^{t,y}(s;\theta_1,\theta_2,u_1,u_2))) \{u_i(x^{t,y}(s;\theta_1,\theta_2,u_1,u_2),\theta_1,\theta_2)\}^2\Big]ds,\ i=1,2 \qquad (6)$$

and

$$\Pi^{t,y}\Big[\theta_1(\cdot),\theta_2(\cdot),u_1(\cdot,\theta_1(\cdot),\theta_2(\cdot)),u_2(\cdot,\theta_1(\cdot),\theta_2(\cdot))\Big] = \int_0^\infty \exp^{-r(s-t)}\Big[(M_1-M_2)x^{t,y}(s;\theta_1,\theta_2,u_1,u_2) + M_2 - \sum_{i=1}^{2}\theta_i(x^{t,y}(s;\theta_1,\theta_2,u_1,u_2))) \{u_i(x^{t,y}(s;\theta_1,\theta_2,u_1,u_2),\theta_1,\theta_2)\}^2\Big]ds, \qquad (7)$$

where we should stress that $\theta_i(\cdot),\ u_i(\cdot,\theta_1(\cdot),\theta_2(\cdot))$ evaluated at any state z are $\theta_i(z),\ u_i(z,\theta_1(z),\theta_2(z))$, $i=1,2$.

We can now define the feedback Stackelberg-Nash equilibrium for the problem.A quadruple of strategies $(\theta_1^*,\theta_2^*,u_1^*,u_2^*) \in \Theta_1\times\Theta_2\times U_1\times U_2$ is called a Stackelberg-Nash equilibrium if the following holds

$$\Pi^{t,y}\Big[\theta_1^*(\cdot),\theta_2^*(\cdot),u_1^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot)),u_2(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot))\Big] \geq \Pi^{t,y}\Big[\theta_1(\cdot),\theta_2(\cdot),u_1^*(\cdot,\theta_1(\cdot),\theta_2(\cdot)),u_2^*(\cdot,\theta_1(\cdot),\theta_2(\cdot))\Big] \quad \forall(\theta_1,\theta_2)\in\Theta_1\times\Theta_2,\quad \forall(t,y)\in[0,\infty)\times[0,1],$$

$$\Pi_1^{t,y}\Big[\theta_1^*(\cdot),\theta_2^*(\cdot),u_1^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot)),u_2^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot))\Big] \geq \Pi_1^{t,y}\Big[\theta_1^*(\cdot),\theta_2^*(\cdot),u_1(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot)),u_2^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot))\Big] \quad \forall u_1\in U_1,\quad \forall(t,y)\in[0,\infty)\times[0,1],$$

$$\Pi_2^{t,y}\Big[\theta_1^*(\cdot),\theta_2^*(\cdot),u_1^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot)),u_2(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot))\Big] \geq \Pi_2^{t,y}\Big[\theta_1^*(\cdot),\theta_2^*(\cdot),u_1^*(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot)),u_2(\cdot,\theta_1^*(\cdot),\theta_2^*(\cdot))\Big] \quad \forall u_2\in U_2,\quad \forall(t,y)\in[0,\infty)\times[0,1].$$

For further details on feedback Stackelberg-Nash equilibrium, see Bensoussan et al. (2019, 2014). Also specified there is a procedure that obtains such an equilibrium. In this paper, we shall use that procedure to solve our problem (2)-(5) to obtain the strategies $\theta_i(x)$ and $u_i(x,\theta_1,\theta_2),\ i=1,2$.

Note that the feedback Stackelberg-Nash equilibrium obtained for the problem (2)-(5), is time consistent, as opposed to the open-loop Stackelberg equilibrium, which, in general, is not; See Bensoussan et al. (2015) for details.

Upon examining the Stackelberg differential game described in equations (2)-(5), we can immediately see that if $M_1=M_2=M\geq 0$, then

$$\int_0^\infty e^{-rt}(M_1x(t)+M_2(1-x(t)))dt = \int_0^\infty e^{-rt}Mdt = M/r,$$

and the objective function of the manufacturer in equation (4) can be written as M/r plus the negative advertising terms inside the integral. It is clear then that the objective is maximized when $\theta_1^*=\theta_2^*=0$, and thus $V(x_0)=M/r$. Before beginning our analysis of the case $M_1>M_2\geq 0$, we note from an inspection of the manufacturer's objective function in equation (4) that only one of the retailers will be supported in all cases.

Proposition 1. *It is never optimal for the manufacturer to support both retailers. In particular,* $M_1>M_2\geq 0$ *implies* $\theta_2^*\equiv 0$.

4 ANALYSIS AND RESULTS

We now begin our study of the case $M_1>M_2\geq 0$. In view of Proposition 1, we set $\theta_2^*\equiv 0$. The next two propositions deal with solving the Stackelberg differential game (2)-(5). For this, let $V_1(x_1),\ V_2(x)$ and $V(x)$ denote the value function of retailer 1, retailer 2 and the manufacturer, respectively. These will be obtained in the course of our analysis.

Proposition 2. *The feedback Stackelberg equilibrium of the game* (2-5) *is characterized as follows:*
(a) *The optimal advertising decisions of the retailers are given by*

$$u_1(x,\theta_1,0) = \frac{V_1'\rho_1\sqrt{1-x}}{2(1-\theta_1)}, u_2(x,\theta_1,0) = -\frac{V_2'\rho_2\sqrt{x}}{2}. \qquad (8)$$

(b) *The optimal participation rate of the manufacturer has the form*

$$\theta_1(x) = \left(\frac{2V'(x)-V_1'(x)}{2V'(x)+V_1'(x)}\right)^+. \qquad (9)$$

(c) *The value functions* V_1*,* V_2*, and* V *for Retailer* 1*, Retailer* 2*, and the manufacturer, respectively, satisfy the following three simultaneous differential equations:*

$$rV_1 = m_1 x + \frac{\rho_1^2 \left(V_1'\right)^2}{4\left(1-\left(\frac{2V'-V_1'}{2V'+V_1'}\right)^+\right)}(1-x) + \frac{V_1'V_2'\rho_2^2}{2}x - V_1'\left(\delta_1 x - \delta_2(1-x)\right) \tag{10}$$

$$rV_2 = m_2(1-x) + \frac{\rho_2^2\left(V_2'\right)^2}{4}x + \frac{V_1'V_2'\rho_1^2}{2\left(1-\left(\frac{2V'-V_1'}{2V'+V_1'}\right)^+\right)}(1-x) - V_2'\left(\delta_1 x - \delta_2(1-x)\right) \tag{11}$$

$$rV = M_1 x + M_2(1-x) - \frac{\rho_1^2\left(V_1'\right)^2\left(\frac{2V'-V_1'}{2V'+V_1'}\right)^+(1-x)}{4\left(\left(\frac{2V'-V_1'}{2V'+V_1'}\right)^+ - 1\right)^2} - \frac{V'V_2'\rho_2^2 x}{2} - \frac{V'V_1'\rho_1^2(1-x)}{2\left(\left(\frac{2V'-V_1'}{2V'+V_1'}\right)^+ - 1\right)} - \delta_1 V' x + \delta_2 V'(1-x) \tag{12}$$

At this point, we conjecture linear value functions, which work for our formulation since it has the square-root feature in the dynamics (5). Specifically, we set $V_1 = \alpha_1 + \beta_1 x$, $V_2 = \alpha_2 + \beta_2(1-x)$, and $V_M = \alpha_M + \beta_M x$ in equations (10-12), where the unknown parameters α_1, β_1, α_2, β_2, α_M, and β_M are constants. Then, by equating the coefficients of x on both sides of equations (10-12), we get six simultaneous algebraic equations that can be solved to obtain the six unknown parameters. The results are provided in Proposition 3.

Proposition 3. (a) *The retailers' optimal advertising decisions are given by*

$$u_1^*(x,\theta_1^*(x),0) = \frac{\beta_1\rho_1\sqrt{1-x}}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)},$$
$$u_2^*(x,\theta_1^*(x),0) = \frac{\beta_2\rho_2\sqrt{x}}{2}. \tag{13}$$

(b) *The optimal participation rate of the manufacturer is a constant given by*

$$\theta_1^*(x) = \left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+. \tag{14}$$

(c) *The value functions of the two retailers and the manufacturer are linear in market share, i.e.,* $V_1 = \alpha_1 + \beta_1 x$*,* $V_2 = \alpha_2 + \beta_2(1-x)$*, and* $V_M = \alpha_M + \beta_M x$*, where the parameters* α_1*,* β_1*,* α_2*,* β_2*,* α_M*, and* β_M *are obtained from the following system of equations:*

$$r\alpha_1 = \frac{\beta_1^2\rho_1^2}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} + \beta_1\delta_2 \tag{15}$$

$$r\beta_1 = m_1 - \frac{\beta_1^2\rho_1^2}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - \frac{\beta_1\beta_2\rho_2^2}{2} - \beta_1(\delta_1+\delta_2) \tag{16}$$

$$r\alpha_2 = \frac{\beta_2^2\rho_2^2}{4} + \beta_2\delta_1, \tag{17}$$

$$r\beta_2 = m_2 - \frac{\beta_2^2\rho_2^2}{4} - \frac{\beta_1\beta_2\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - \beta_2(\delta_1+\delta_2), \tag{18}$$

$$r\alpha_M = M_2 - \frac{\beta_1^2\rho_1^2\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)^2} + \frac{\beta_1\beta_M\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} + \delta_2\beta_M, \tag{19}$$

$$r\beta_M = M_1 - M_2 + \frac{\beta_1^2\rho_1^2\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)^2} - \frac{\beta_2\beta_M\rho_2^2}{2} - \frac{\beta_1\beta_M\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - (\delta_1+\delta_2)\beta_M, \tag{20}$$

There are now two cases to consider: $\theta_1^* = 0$ and $\theta_1^* > 0$.

4.1 Case $\theta_1^* = 0$

Whenever this case arises, and we will, later in Proposition 4, determine the precise conditions on the problem parameters when it does, we must have the value-function coefficients that satisfy the condition $\frac{2\beta_M-\beta_1}{2\beta_m+\beta_1} \leq 0$, and these coefficients must solve the following system of equations obtained from (15-20) by setting $\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+ = 0$:

$$r\alpha_1 = \frac{\beta_1^2\rho_1^2}{4} + \beta_1\delta_2, \tag{21}$$

$$r\beta_1 = m_1 - \frac{\beta_1^2\rho_1^2}{4} - \frac{\beta_1\beta_2\rho_2^2}{2} - \beta_1(\delta_1+\delta_2), \tag{22}$$

$$r\alpha_2 = \frac{\beta_2^2\rho_2^2}{4} + \beta_2\delta_1, \tag{23}$$

$$r\beta_2 = m_2 - \frac{\beta_2^2\rho_2^2}{4} - \frac{\beta_1\beta_2\rho_1^2}{2} - \beta_2(\delta_1+\delta_2), \tag{24}$$

$$r\alpha_M = M_2 + \frac{1}{2}\beta_1\beta_M\rho_1^2 + \beta_M\delta_2, \tag{25}$$

$$r\beta_M = M_1 - M_2 - \frac{1}{2}\beta_1\beta_M\rho_1^2 - \frac{1}{2}\beta_2\beta_M\rho_2^2 - \beta_M(\delta_1+\delta_2). \tag{26}$$

Given these equations, we can state the following result that presents the *participation threshold*, i.e., the point at which the manufacturer moves from $\theta_1^* = 0$ to $\theta_1^* > 0$.

Proposition 4. *The participation threshold S is given by*

$$S = 2(M_1 - M_2) - m_1 - \frac{\beta_1^2\rho_1^2}{4}, \tag{27}$$

where β_1 is the unique positive solution of

$$\beta_1^4 + \kappa_1\beta_1^3 + \kappa_2\beta_1^2 - \kappa_3 = 0 \tag{28}$$

with $\kappa_1 = \frac{16(r+\delta_1+\delta_2)}{3\rho_1^4}$,
$\kappa_2 = \frac{16(r+\delta_1+\delta_2)^2 - 8m_1\rho_1^2 + 16m_2\rho_2^2}{3\rho_1^4}$, *and* $\kappa_3 = \frac{16m_1^2}{3\rho_1^4}$. *The manufacturer chooses $\theta_1^* > 0$ when $S > 0$ and $\theta_1^* = 0$ when $S \leq 0$. Furthermore, $\partial S/\partial(M_1 - M_2) = 2 > 0$ at $S \leq 0$.*

Note that S is computed by solving the system of equations in (21-26) for $\theta_1^* = 0$, in which case $S \leq 0$. From the result $\partial S/\partial(M_1 - M_2) > 0$ at $S = 0$, we can immediately see that when $S = 0$, a decrease in $(M_1 - M_2)$ will keep $\theta_1^* = 0$, whereas an increase in $(M_1 - M_2)$ will make it optimal for the manufacturer to set $\theta_1^* > 0$. This is because an increase in the difference in the manufacturer's margins from Retailers 1 and 2 offers the manufacturer a greater incentive to provide promotional support to Retailer 1.

Next, we state Corollary 1, which provides the solutions to equations (21-26) for the case when the two retailers are symmetric, i.e., $\delta_i = \delta,\ \rho_i = \rho,\ r_i = r,\ m_i = m,\ \alpha_i = \alpha$, and $\beta_i = \beta,\ i = 1,2$.

Corollary 1.
(a) *The value-function coefficients of the two retailers in the symmetric case are given by*

$$\alpha = \frac{4m\rho^2 - (r-\delta)\left(\sqrt{(r+2\delta)^2 + 3m\rho^2} - (r+2\delta)\right)}{9r\rho^2/2},$$

$$\beta = \frac{\sqrt{(r+2\delta)^2 + 3m\rho^2} - (r+2\delta)}{3\rho^2/2}. \tag{29}$$

(b) *The value-function coefficients of the manufacturer are given by*

$$\alpha_M = \frac{2rM_2 + \left(\beta\rho^2 + 2\delta\right)(M_1 + M_2)}{2r\left(\beta\rho^2 + r + 2\delta\right)},$$

$$\beta_M = \frac{M_1 - M_2}{\beta\rho^2 + r + 2\delta}, \tag{30}$$

where β is specified by (29).
(c) *We have the following comparative statics results for S w.r.t. the model parameters: $\partial S/\partial(M_1 - M_2) = 2 > 0$, $\partial S/\partial m < 0$, $\partial S/\partial\rho < 0$, $\partial S/\partial r > 0$, $\partial S/\partial\delta > 0$.*
(d) *At the manifold $S = 0$, we have the following comparative statics results for the model parameters: $\partial(M_1 - M_2)/\partial m > 0$, $\partial\rho/\partial m < 0$, $\partial(M_1 - M_2)/\partial\delta < 0$, $\partial m/\partial\delta > 0$, $\partial\rho/\partial\delta > 0$, $\partial r/\partial\delta = -2 < 0$.*

Corollary 2. *When $r = \delta = 0$, S in* (27) *reduces to*

$$S = \frac{1}{\sqrt{3m\rho^2}}(3(M_1 - M_2) - 2m). \tag{31}$$

Thus, S in (31) can be used as an approximate participation threshold when r and δ are small compared to m, M_1, M_2, and ρ.

4.2 Case $\theta_1^* > 0$

When the solution in Section 4.1 results in $S > 0$, we know $\theta_1^* > 0$. Then, by substituting $\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+ = \frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}$ into equations (15-20), we have the following system of equations to solve for the value-function coefficients:

$$r\alpha_1 = \frac{1}{8}\beta_1\rho_1^2(2\beta_M + \beta_1) + \beta_1\delta_2, \tag{32}$$

$$r\alpha_2 = \frac{1}{4}\beta_2^2\rho_2^2 + \beta_2\delta_1, \tag{33}$$

$$r\alpha_M = M_2 + \frac{1}{16}\rho_1^2(2\beta_M + \beta_1)^2 + \beta_M\delta_2, \tag{34}$$

$$r\beta_1 = m_1 - \frac{1}{8}\beta_1\rho_1^2(2\beta_M + \beta_1) - \frac{1}{2}\beta_1\beta_2\rho_2^2 - \beta_1(\delta_1 + \delta_2) \tag{35}$$

$$r\beta_2 = m_2 - \frac{1}{4}\beta_2\rho_1^2(2\beta_M+\beta_1) - \frac{1}{4}\beta_2^2\rho_2^2 - \beta_2(\delta_1+\delta_2), \tag{36}$$

$$r\beta_M = M_1 - M_2 - \frac{1}{16}\rho_1^2(2\beta_M+\beta_1)^2 - \frac{1}{2}\beta_2\beta_M\rho_2^2 - \beta_M(\delta_1+\delta_2). \tag{37}$$

This requires numerical analysis. Figure 1 illustrates the effect of the retailer's margin on the manufacturer's participation rate. We use the following sample parameter values: $r = 0.03$, $\rho_1 = \rho_2 = 0.5$, $\delta_1 = 0.07$, $\delta_2 = 0.1$, $M_1 = 0.5$, $M_2 = 0.2$. Furthermore, for the effect of Retailer 1's margin, we set Retailer 2's margin to $m_2 = 0.5$, and vice versa.

We already know from Corollary 1(c) that $\partial S/\partial m < 0$ at $S \leq 0$ in the symmetric case. The result shown in Figure 1—that θ_1^* decreases, and at a deceasing rate, as the margin m_1 of the supported retailer increases—is consistent with our finding in the symmetric case and represents its generalization to the asymmetric case. The reason behind this result is that

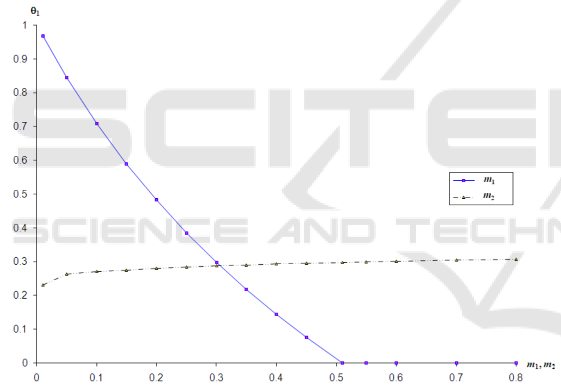

Figure 1: Effect of Retailers' Margins on the Optimal Participation Rate.

with low margin m_1, Retailer 1 will under-advertise and the manufacturer's profit will suffer. So, then, it is in the interest of the manufacturer to participate so as to encourage the retailer to advertise more. As m_1 increases, Retailer 1 has its own incentive to advertise, and, therefore, the manufacturer does not need to offer as much in the way of participation. As m_1 keeps increasing, the manufacturer ceases to participate altogether. We see from Figure 1 that θ_1^* indeed becomes zero at $m_1 \approx 0.5$, where the switch from "cooperative advertising" to "no cooperative advertising" takes place. A more precise value can be obtained by using (27), and we find this to be $m_1 = 0.50925$. Thus, we have generalized to the competitive environment the result obtained in He et al. (2009) in the absence of competition.

The effect of the margin of the non-supported retailer, i.e., Retailer 2, on the offer of cooperative advertising to Retailer 1, the supported retailer, appears to be weak. Note that at $m_1 = 0.3$, the two curves in Figure 1 cross. At this point, as m_1 increases *ceteris paribus*, θ_1^* decreases sharply whereas as m_2 increases, θ_1^* increases but slowly. This means that as m_2 increases, Retailer 1 faces greater advertising competition from the competing retailer and this induces the manufacturer to support Retailer 1 at a slightly higher rate. On the other hand, as m_1 increases, as has already been mentioned, it increases the incentive of Retailer 1 to advertise of its own accord, and thus the manufacturer reduces its support significantly. Even though m_2 has a weak effect on θ_1^*, it may still make a difference between the manufacturer supporting or not supporting Retailer 1. Numerical analysis shows that when $m_1 = m_2 = 0.5$, then $\theta_1^* = 0.012$. But as m_2 decreases from 0.5 (e.g., $m_2 = 0.25$), the manufacturer stops offering promotional support to Retailer 1. On the other hand, as m_2 increases from 0.5 to 0.80, θ_1^* increases from a support of 1.2% to only 3.2%.

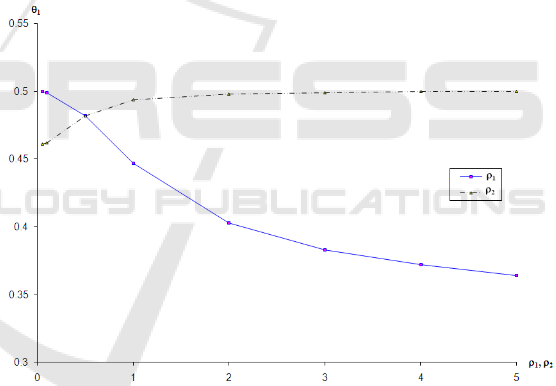

Figure 2: Effect of Advertising Effectiveness on the Optimal Participation Rate.

In Figure 2, the fixed parameter values are $m_1 = 0.2$, $m_2 = 0.5$, $r = 0.03$, $\rho_1 = 0.5$, $\delta_1 = 0.07$, $\delta_2 = 0.1$, $M_1 = 0.5$, $M_2 = 0.2$. Consistent with the comparative statics result for the symmetric case in Corollary 1(c) that $\partial S/\partial\rho < 0$, this figure shows that as the advertising effectiveness of the supported retailer increases, the degree of support by the manufacturer diminishes. The reason is that, given the greater effectiveness of advertising, the retailer has an incentive to advertise at a higher level even without the support of the manufacturer. The unsupported retailer's advertising effectiveness does not have a great impact, but as it increases, it slowly raises the participation rate.

Finally, Figure 3 shows that as the sum of the market share decay rates for the two retailers increases,

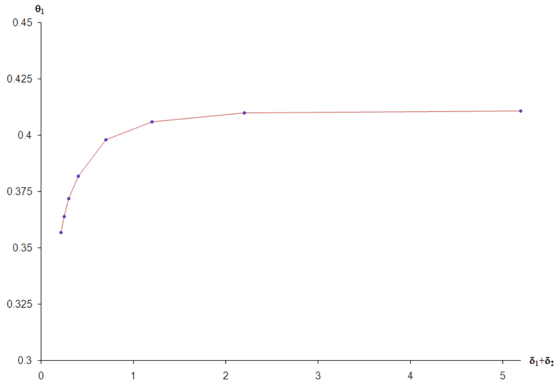

Figure 3: Effect of Decay on the Optimal Participation Rate.

the participation rate to the supported retailer, i.e., Retailer 1, increases. For this figure, the remaining parameter values are $m_1 = 0.5$, $m_2 = 0.2$, $r = 0.03$, $\rho_1 = 0.5$, $\rho_2 = 0.5$, $M_1 = 0.8$, $M_2 = 0.2$. Note that we have plotted θ_1^* against the sum of the decay rates since it is clear from the dynamics in (1) that the decay rate for Retailer 1 is $(\delta_1 + \delta_2)$. It is for this reason that we can easily see from equations (32-37) that β_1, β_2, and β_M, and, therefore, θ_1^*, are affected by the sum $(\delta_1 + \delta_2)$ and not by the decay rates individually. The result can be explained intuitively since the increase of $(\delta_1 + \delta_2)$ is similar to increasing the speed of the treadmill with which the advertising must keep up. Thus, as $(\delta_1 + \delta_2)$ increases, Retailer 1 finds it more expensive to maintain its market share and the manufacturer must offer a higher participation rate to Retailer 1 to adequately promote the product. This finding generalizes to the asymmetric case the analytical result obtained for the symmetric case in Corollary 1(c) that $\partial S/\partial\delta < 0$. Finally, note that the effect of the decay rates is most pronounced at their lower values and does not have much effect on the participation rate at higher values.

5 EXTENSIONS

The model can be extended to include retail price competition. In the case where the retail price, denoted p_i, is endogenously determined in addition to the advertising decision u_i, Retailer i's discounted profit maximization problem is given by

$$V_i(x_0) = \max_{u_i(t),\, p_i(t)} \int_0^\infty e^{-rt}\Omega dt \quad i = 1,2, \tag{38}$$

subject to (1), where

$$\Omega = D_i(p_i(t), p_{3-i}(t))(p_i(t) - c_i)x_i(t) - (1 - \theta_i(x(t)))u_i^2(t),$$

$p_i(t)$ is the price charged by Retailer i,

$$D_i(p_i(t), p_{3-i}(t)) = (1 - b_i p_i(t) + d_{3-i} p_{3-i}(t))$$

is the demand function expressed as a function of the prices of the two retailers, c_i is the marginal cost of Retailer i, and b_i and d_i are demand parameters. Solving for the retailers' optimal prices yields

$$\hat{p}_i = \frac{d_{3-i} + b_{3-i}(2 + 2b_i c_i + d_{3-i} c_{3-i})}{4b_1 b_2 - d_1 d_2}, \quad i = 1,2. \tag{39}$$

Using the parameter m_i to denote $(D_i(p_i(t), p_{3-i}(t)))(p_i(t) - c_i)$ in equation (38), where $p_i = \hat{p}_i$ from equation (39), results in equations (2-3).

The demand parameters affect price, which, in turn, affects the profit margins. Thus, the results in Section 4 remain unchanged except that we can now add that the demand parameters affect the participation rate θ_1^* via m_i, $i = 1, 2$.

As in Prasad and Sethi (2004), one could extend the model to include a stochastic term in the dynamic equation in (1) as follows:

$$\begin{aligned} dx(t) = & \left(\rho_1 u_1(t)\sqrt{1 - x(t)} - \rho_2 u_2(t)\sqrt{x(t)}\right)dt \\ & + (-\delta_1 x(t) + \delta_2(1 - x(t)))dt \\ & + \sigma(x(t))dz(t) \end{aligned} \tag{40}$$

where $\sigma(x(t))$ represents a variance term and $z(t)$, $t \geq 0$, is the standard Wiener process.

6 CONCLUSIONS

In this paper, we consider a manufacturer who sells a product to one or both of two independent and competing retailers. The retailers invest in local advertising effort, while the manufacturer decides whether or not to support the retailers' advertising activities through a participation rate. We use differential game theory to solve the Stackelberg game between the manufacturer and the two retailers.

We derive the optimal advertising levels of the two retailers and the participation rates of the manufacturer, and find that the manufacturer will provide strictly less than 100% participation rate, and that the manufacturer will offer a non-zero participation rate to the retailer from whom the manufacturer earns the higher margin. We then analyzed several special cases of the general model, including the case of symmetric retailers. Compared to He et al. (2009), who do not model retail competition, we find that the presence of a competing retailer induces the manufacturer to provide a higher level of cooperative advertising support to its retailer, and for a greater range of parameter values, than if that retailer were a monopolist.

There remain open issues for future research. To preserve tractability, we did not model the manufacturer's wholesale price decision in this paper, as was done in He et al. (2009). Future research could extend the model to also include the manufacturer's wholesale price. It would also be interesting to explore decisions in a retail oligopoly, along the lines of Fruchter (1999).

Finally, this paper opens up a fruitful avenue for future empirical research. Since we have provided the optimal participation rates as well as their dependence on various firm- and industry-level parameters, it would now be possible to empirically examine whether our results can explain the participation rates in different industries as reported in Dutta et al. (1995). This would of course require estimation of the firm- and industry-level parameters, possibly employing the techniques used in Naik et al. (2008). Provided an appropriate data set can be found or collected, an empirical study to validate our results would undoubtedly deepen our understanding of the cooperative advertising practices in different industries.

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APPENDIX

Proof of Proposition 1

First, note that if $M_1 > M_2$, then the manufacturer's objective function in equation (4) can be organized as follows: $\int_0^{\infty} e^{-rt}\left(M_2+(M_1-M_2)x-\theta_1 u_1^2(x)-\theta_2 u_2^2(x)\right)dt$. This suggests that the manufacturer should encourage Retailer 1 to increase its advertising to increase x (and discourage Retailer 2 to decrease advertising and $(1-x)$ by setting $\theta_2^*=0$) as much as possible. *xspace* ■

Proof of Proposition 2

The Hamilton-Jacobi-Bellman (HJB) equation (Sethi and Thompson 2000) for Retailer i is given by:

$$rV_1 = \max_{u_1}\Big\{m_1 x-(1-\theta_1)u_1^2+V_1'\left(\rho_1 u_1\sqrt{1-x}\right) -V_1'\left(\rho_2 u_2\sqrt{x}-\delta_1 x+\delta_2(1-x)\right)\Big\}, \quad (41)$$

$$rV_2 = \max_{u_2}\Big\{m_2(1-x)-(1-\theta_2)u_2^2 +V_2'\left(\rho_1 u_1\sqrt{1-x}\right) -V_2'\left(\rho_2 u_2\sqrt{x}-\delta_1 x+\delta_2(1-x)\right)\Big\} \quad (42)$$

The first-order conditions for maximization yield the optimal advertising levels in equation (8) in Proposition 2(a). Substituting these solutions into the above HJB equations for the two retailers yields equations (10-11) in Proposition 2(c).

The HJB equation for the manufacturer is given by:

$$rV = \max_{\theta_1,\,\theta_2}\begin{cases} M_1x+M_2(1-x)-\theta_1 u_1^2(x|\theta_1,\theta_2) \\ -\theta_2 u_2^2(x|\theta_1,\theta_2) \\ +V'\left(\rho_1 u_1(x|\theta_1,\theta_2)\sqrt{1-x}\right) \\ -V'\left(\rho_2 u_2(x|\theta_1,\theta_2)\sqrt{x}-\delta_1 x+\delta_2(1-x)\right). \end{cases} \quad (43)$$

Substituting the optimal advertising efforts of the two retailers into the above equation and simplifying yields

$$rV = \max_{\theta_1,\,\theta_2}\begin{cases} M_1x+M_2(1-x)-\frac{1}{4}\left(\frac{\theta_1\rho_1^2\left(V_1'\right)^2(1-x)}{(1-\theta_1)^2}+\frac{\theta_2\rho_2^2\left(V_2'\right)^2 x}{(1-\theta_2)^2}\right) \\ -\left(\frac{(1-\theta_1)\rho_2^2 V_2' x+(1-\theta_2)\left(2(\delta_1 x-\delta_2(1-x))(1-\theta_1)-\rho_1^2 V_1'(1-x)\right)}{2(1-\theta_1)(1-\theta_2)}\right)V'. \end{cases} \quad (44)$$

Solving the first-order conditions for the participation rates, and noting that $M_1 > M_2$ implies $\theta_2^*=0$, yields equation (9) in Proposition 2(b). Substituting these solutions into equation (44) and simplifying, we obtain equation (12) in Proposition 2(c). *xspace* ■

Proof of Proposition 3

With $V_1=\alpha_1+\beta_1 x$ and $V_2=\alpha_2+\beta_2(1-x)$, we have $V_1'=\beta_1$ and $V_2'=\beta_2$. Inserting these into (10) and (11), we have

$$r(\alpha_1+\beta_1 x)=m_1 x+\frac{\beta_1^2\rho_1^2(1-x)}{4\left(1-\left(\frac{2V'-\beta_1}{2V'+\beta_1}\right)^+\right)} -\frac{\beta_1\beta_2\rho_2^2 x}{2}-\beta_1(\delta_1 x-\delta_2(1-x)), \quad (45)$$

$$r(\alpha_2+\beta_2(1-x))=m_2(1-x)+\frac{\beta_2^2\rho_2^2 x}{4} -\frac{\beta_1\beta_2\rho_1^2(1-x)}{2\left(1-\left(\frac{2V'-\beta_1}{2V'+\beta_1}\right)^+\right)}+\beta_2(\delta_1 x-\delta_2(1-x)). \quad (46)$$

With $V=\alpha_M+\beta_M x$, we have $V'=\beta_M$. Substitution into equations (8) and (9) yields (13) in Proposition 3(a) and (14) in Proposition 3(b), respectively. Substituting these into the HJB equation in (12) and simplifying yields

$$r(\alpha_M+\beta_M x) = M_1x+M_2(1-x)-\frac{\beta_2\beta_M\rho_2^2 x}{2} -\frac{\beta_1^2\rho_1^2\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+(1-x)}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)^2} -\frac{\beta_1\beta_M\rho_1^2(1-x)}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} -\delta_1\beta_M x+\delta_2\beta_M(1-x). \quad (47)$$

Equating the powers of x in (47), we obtain equations (19-20) in Proposition 3(c). Substituting $V' = \beta_M$ into (45-46) and equating the powers of x and $(1-x)$ yields equations (15-18) in Proposition 3(c). *xspace* ■

Proof of Proposition 4

Solving equation (22) for β_2 yields

$$\beta_2 = \frac{4m_1 - \beta_1\left(\beta_1\rho_1^2 + 4(r+\delta_1+\delta_2)\right)}{2\beta_1\rho_2^2}. \quad (48)$$

Substituting the above solution into equation (24) and simplifying yields the quartic equation (28).

As detailed in Prasad and Sethi (2004), it can be shown that there exists a unique $\beta_1 > 0$ that solves the above quartic equation. This is the unique equilibrium of the Stackelberg differential game. That solution can then be substituted into (48) to yield the solution for β_2. We know from (26) that

$$\beta_M = \frac{2(M_1 - M_2)}{\beta_1\rho_1^2 + \beta_2\rho_2^2 + 2(r+\delta_1+\delta_2)}. \quad (49)$$

Substituting the solutions for β_1 and β_2 into (49) yields β_M.

Since $\beta_M > 0$ from equation (20), the participation threshold is obtained as $2\beta_M - \beta_1$ as follows;

$$\frac{4(M_1-M_2) - \beta_1^2\rho_1^2 - \beta_1\beta_2\rho_2^2 - 2\beta_1(r+\delta_1+\delta_2)}{\beta_1\rho_1^2 + \beta_2\rho_2^2 + 2(r+\delta_1+\delta_2)}. \quad (50)$$

We know from equation (22) that

$$\beta_1\beta_2\rho_2^2 + 2\beta_1(\delta_1+\delta_2) = \frac{1}{2}\left(4m_1 - 4r\beta_1 - \beta_1^2\rho_1^2\right).$$

Using this in the numerator of (50) and simplifying gives the participation threshold S in (27).

Since β_1 is independent of $(M_1 - M_2)$, we have $\partial S/\partial(M_1 - M_2) = 2 > 0$. *xspace* ■

Proof of Corollary 1

The solutions for α_i and β_i, $i = 1,2$, in Corollary 1(a), are the same as in the symmetric analysis in Prasad and Sethi (2004) in the absence of uncertainty.

For the manufacturer, as before, the linear value function $V = \alpha_M + \beta_M x$ solves the HJB equation. Setting $\theta_1 = \theta_2 = 0$, imposing symmetry in the model parameters, and simplifying the HJB equation results in

$$r(\alpha_M + \beta_M x) = M_2 + \frac{\beta_M\left(\beta\rho^2 + 2\delta\right)}{2} + \left(M_1 - M_2 - \left(\frac{\beta\rho^2 + 4\delta}{2}\right)\beta_M\right)x. \quad (51)$$

Equating the coefficients of x in equation (51) yields the solutions in equation (30) in Corollary 1(b). Note that this can be interpreted as $\int_0^\infty e^{-rt}\left(M_1 x + M_2(1-x)\right)dt$ given the solution obtained with $\theta_1 = \theta_2 = 0$.

For the comparative statics, imposing symmetry in the participation threshold from equation (27), we have $S(.) = 2(M_1 - M_2) - m - \frac{\beta^2\rho^2}{4} = 0$. Substituting the solution for β from equation (29), this can be rewritten as

$$S(.) = 2(M_1 - M_2) - \frac{4}{3}m + \frac{2(r+2\delta)\left(\sqrt{(r+2\delta)^2 + 3m\rho^2} - (r+2\delta)\right)}{9\rho^2}. \quad (52)$$

Taking the derivative of S in equation (52) w.r.t. the model parameters yields the comparative statics in Corollary 1(c): $\partial S/\partial(M_1 - M_2) > 0$, $\partial S/\partial m < 0$, $\partial S/\partial\rho < 0$, $\partial S/\partial r > 0$, $\partial S/\partial\delta > 0$.

Proof of Corollary 2

We know that $V' = \beta_M$ and $\beta_1 = \beta_2 = \beta$, where $\beta = \frac{\sqrt{(r+2\delta)^2+3m\rho^2}-(r+2\delta)}{3\rho^2/2}$. Moreover, $\beta_M = \frac{M_1 - M_2}{\beta\rho^2 + r + 2\delta}$, with the aforementioned β.

Since $\theta_1 = \frac{2V' - \beta_1}{2V' + \beta_1} = \frac{2\beta_M - \beta}{2\beta_M + \beta}$, we need $2\beta_M - \beta > 0$ for $\theta_1 > 0$. Substituting the value of β into β_M and simplifying $2\beta_M - \beta$, we have

$$S = 2\beta_M - \beta = \frac{6(M_1 - M_2)}{r + 2\delta + 2\sqrt{(r+2\delta)^2 + 3m\rho^2}} - \frac{2\left(\sqrt{3m\rho^2 + (r+2\delta)^2} - (r+2\delta)\right)}{3\rho^2}. \quad (53)$$

Therefore, $\theta_1^* > 0$ when $S > 0$ from above. With $r = \delta = 0$ in (53), we obtain equation (31) as the approximation of (53). *xspace* ■