Introducing Conics in 9th Grade: An Experimental Teaching

Borislav Yordanov Lazarov[1] and Dimitar Georgiev Dimitrov[2]
[1]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Blok 8 Georgi Bonchev Str., Sofia, Bulgaria
[2]125th Secondary School, Sofia, Bulgaria

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Abstract: Under consideration is an experimental teaching focused on presenting the conics in a dual manner: as loci and envelopes. A bunch of computer technologies is drawn to explore and investigate this duality of the conics. An example of how it is done for a particular conic is given. The target group includes secondary school students who are advanced in math and information technologies. The theoretical base is an original didactical model for designing individual educational trajectories that is adapted for the team-working mode. The educational goal includes developing synthetic competence of an entire team. The individual characteristics of the team members complement one another for resolving complex problems from the local behavioral environment, which were specifically formed for the purposes of the experimental teaching.

1 INTRODUCTION

The new Bulgarian educational legislation allows a type of schools (so-called *innovative schools*) to vary the syllabus including integrated subjects into the school plan. Such status quo provides the opportunity for creative teaching. However, there are no textbooks neither educational plans nor guidelines for the new subjects. Therefore, any good practice is welcome in this slippery twilight zone. Below we are going to share our experience in teaching conics in 9th grade, hoping it is a kind of good practice.

The approach we applied integrates areas of several types.

1) Organizational: it is an example of integrating school and academic staff.
2) Technological: it integrates classical geometry with computer technologies.
3) Didactical: we combine the deductive with algorithmic method in math.
4) Mathematical: we synthesized the concepts of locus and envelope by a common algorithm for any of the conics.

The experimental teaching took part in 2018/2019 scholastic year with a team of three ninthgraders, who are advanced both in math and ICT. Further we refer to this team as *the Team*. The small numbers of the students in the Team allowed to apply individual approach on one hand, and to take advantages of the team work on the other hand. The didactical model we applied is a modification of DMT (Lazarov, 2013) for indiviual educational trajectory of small team. Some details of the model are given in (Lazarov, 2019).

2 CHOOSING THE TOPIC

We carefully selected the topic for our experimental teaching to satisfy two controversial requirements: the matter to be in the zone of proximal development and to be beyond the curriculum. Our previous practice in teaching conics is presented in (Lazarov and Todorova, 2014), where parabola was introduced in an inquiry based mode of classroom activity. The modest results of this experiment was related with the attempt to organize Socratic style teaching in a large group of students. Nevertheless, some of them made considerable progress, which gave us reasons to reconsider the topic, this time applying individual approach.

The concept of locus used to be a part of the 8th grade Bulgarian curriculum until 2016. Nowadays it remains just a trace of this concept in the 7th grade curriculum. The loci are included implicitly in the properties of the segment bisector and the angle bisector. Fortunately, these two loci was just we need as a base for developing the topic. The concept of envelope of one parametric family of lines was

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completely new matter for the Team (it is rather far from the secondary school mathematics). The modern dynamic geometry software (DGS) provides some opportunities to elaborate the envelope in pure geometrical mode, which is acceptable for the advanced school students (Lazarov, 2011). However, some concepts, which are routine for the calculus, should be reconsidered from the secondary school viewpoint.

3 ABOUT THE DEFINITIONS

The proper selection of definitions is another challenge we met. A variety of didactical approaches could be used for introducing a math concept: by examples and counterexamples, constructively, deductively etc. Our key reason was to stay as close to the curriculum as possible.

3.1 Basic Concepts

In our opinion, the closest to the 9th grade math curriculum are the following definitions:

- Parabola is the locus of points in the plane that are equidistant from a given point (focus) and a given line (directrix).
- Ellipse is the locus of points for which the sum of the distances to two given points (foci) is constant.
- Hyperbola is the locus of points for which the absolute value of the difference of the distances to two given points (foci) is constant.

The 'is constant' in the last two bullets means 'equals a given segment'. These definitions also match the GeoGebra operators for drawing the conics (GeoGebra, 2019). In fact, the GeoGebra operators require a third point for constructing the ellipse and hyperbola (as say the operator icons). The 'given segment' appears as a sum or difference of the distances from this third point to the foci.

3.2 Calculus-circumventing

The calculus definition of tangent line to a curve is inapplicable in 9th grade. We defined the *tangency* in a specific manner for any particular conic. A line is tangent to:

- parabola, iff it has exactly one common point with it and is not perpendicular to its directrix;
- ellipse, iff it has exactly one common point with it;
- hyperbola, iff it has exactly one common point with it and is not parallel to any of its asymptotes.

These definitions extended the familiar concept of tangent line to a circle.

Further, we need to elaborate into geometry the concept of parameter that is common for algebra. The parameter was introduced as:

- point at a straight line when consider parabola;
- point at a circle when consider ellipse;
- point at two circle arcs when consider hyperbola.

We considered only one-parameter families of straight lines, which allowed us to restrict the notion of *envelope of a one-parametric family of lines*. It is a conic, which touches each line from the family in a single point and any line from the family is tangent to this conic.

So, we managed to avoid the concepts from the calculus and to stay inside the secondary school mathematics. Let us note that a similar approach is adopted in (Lazarov, 2011) (but due to the larger generality of considerations in that work, a need to introduce the concept of tangent curves appears).

4 THE CONSTRUCTIONS

Three algorithms for constructing conics were the core of the experimental teaching. Geometrical constructions are included in 7th and 8th grade of the Bulgarian curriculum. Among the conics, only parabola is studied but just as the graph of quadratic function and with no properties of the curve itself.

We decide to develop a constructive method to unify in one algorithm the dual nature of a conic: as locus on one hand and as envelope on the other hand. Here it is the algorithm for the hyperbola H having foci $F_{1,2}$ and passing through a given point T:

1) Denote $c = |F_1T - F_2T|$ and draw the circle k with radius c centred at F_1.
2) Take an arbitrary point $K \in k$.
3) Draw the segment bisector s_{F_2K} of F_2K (Figure 1).

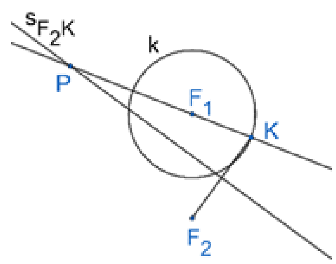


Figure 1: The hyperbola construction algorithm.

***Lemma* 1.** When it exists, the intersection point P of F_1K and s_{F_2K} is at H.

(The proofs of this lemma and the statements that follow are given in the Appendix.)

***Lemma* 2.** If P exists then s_{F_2K} is the tangent to H at P.

Parametrizing the algorithm by K, we can obtain both objects: points at the hyperbola H (according to Lemma 1) and tangents to it (according to Lemma 2). Let us note that P exists iff F_2K is not a tangent to k. The tangency of F_2K and k happens in two points $K_{1,2}$ that split k into two arcs $k_{1,2}$. So, we can take these two arcs as the domain of K. In the case, when K runs along $k_1 \cup k_2$, we obtain a one-parametric-family of lines $\mathcal{H} = \{s_{F_2K}: K \in k_1 \cup k_2\}$. (The lines $s_{F_2K_{1,2}}$ are the asymptotes of H.)

***Theorem* 1.** H is the envelope of $\mathcal{H}$

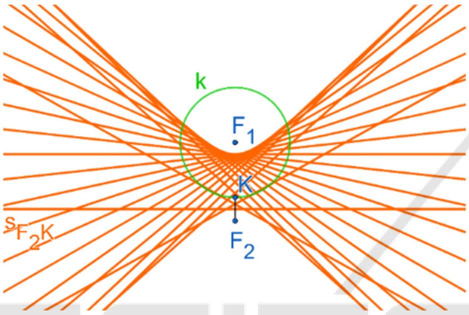


Figure 2: The hyperbola as the envelope of $\mathcal{H}$.

Thus, the hyperbola H occurs in the same time the locus of P and the envelope of s_{F_2K}. Similar constructions were done for parabola and ellipse. In fact, the construction algorithm for the parabola (Lazarov, 2011) serves as a template for the ellipse and hyperbola algorithms, which were done by the Team. It is clear that the algorithms are device-independent: any one could be realized either as a DGS applet or as a traditional ruler-and-compass construction.

5 DIVIDENDS

The unified algorithmic introduction of the conics allows an easy explanation of the reflective properties of these curves. Let us note that according to the laws of the geometric optics a smooth curve reflects an incoming ray as it does the tangent line at the reflection point. Having a conic as locus of points, one operate with a 'real object'. The envelope configures the conic as a phantom, but this phantom performs the reflection of a beam: any ray is reflected by the corresponding tangent line. Combining the dual nature of the conic, one gets the big picture of the reflection. We will illustrate this again with the hyperbola.

***Theorem* 2.** A ray coming from inside of one of the branches of H and directed to the focus inside the other branch, after reflection by H, passes through the focus inside the first branch (Figure 3).

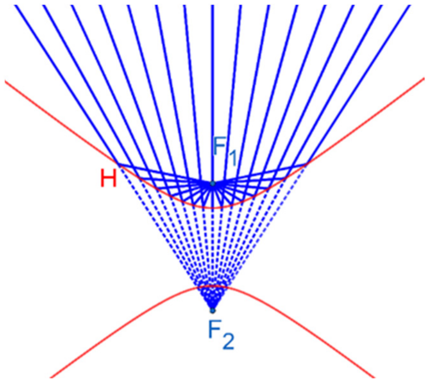


Figure 3: The reflective property of the hyperbola.

Scholium. The reflective property of the hyperbola could work in the reverse manner of the one that is described in Theorem 3: in Figure 3 the sound coming from F_1, after reflection by H, is uniformly distributed inside the branch, which contains F_1. In our opinion, Netrebko and Garanca (2007) use this property when they sing facing the wall. It is also applied to improve the acoustics of the cathedral La Sagrada Familia in Barcelona (Burry et al., 2011). Connecting mathematical results with high achievements in arts significantly lifts the students' attitude to math (Lazarov, 2019).

6 TEAM EDUCATIONAL TRAJECTORY

We followed the general structure of the model DMT (Lazarov, 2013, 2019) for the experimental teaching, which consists of iterative steps (climbing floors). However, the design of the individual educational trajectory (IET) needed slight modification taking into account the students' interaction inside the Team.

In our general educational plan, the proximal educational goals were stated to the Team but any student was in charge for some details. Nevertheless, the elaboration of the details happened in collaboration of the team members and as result a team product appears when the particular educational goal was reached.

6.1 On the First Floor of the IET

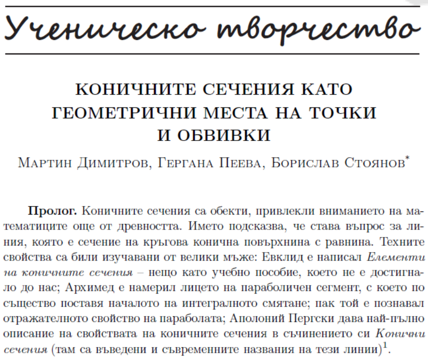
We started the experimental teaching during the summer holyday. The students were proposed to examine closely GeoGebra by themselves and to get an idea about TeX.

During the first term of the scholastic year, we had lectures once weekly. The introductory part of the teaching we devoted to the basic facts about the conic sections. Initially their properties were examined experimentally but then were rigorously proven. Meanwhile, students mastered their skills in GeoGebra and TeX.

6.2 On the Second Floor of the IET

The proximal educational goal after introductory part was to apply the knowledge about conics and the DGS skills for modifying the parabola module to similar modules about ellipse and hyperbola. These activities were directed to prepare a paper for the Bulgarian national math journal for school students *Matematika*.

The Team was motivated and the students did their best to polish all details. E.g., 39 GeoGebra applets were made and there were done several interim variants for the most of them. The paper also needed several redactions. These efforts paid themselves: the paper was published (Dimitrov et al., 2019). Let us note that students' papers were not published in the journal for a rather long time before. This paper resurrected the column *Students' works* (in Bulgarian „Ученическо творчество" – Figure 4).


Ученическо творчество

КОНИЧНИТЕ СЕЧЕНИЯ КАТО ГЕОМЕТРИЧНИ МЕСТА НА ТОЧКИ И ОБВИВКИ

Мартин Димитров, Гергана Пеева, Борислав Стоянов*

Пролог. Коничните сечения са обекти, привлекли вниманието на математиците още от древността. Името подсказва, че става въпрос за линия, която е сечение на кръгова конична повърхнина с равнина. Техните свойства са били изучавани от велики мъже: Евклид е написал *Елементи на коничните сечения* – нещо като учебно пособие, което не е достигнало до нас; Архимед е намерил лицето на параболичен сегмент, с което по същество поставя началото на интегралното смятане; пак той е познавал отражателното свойство на параболата; Аполоний Пергски дава най-пълно описание на свойствата на коничните сечения в съчинението си *Конични сечения* (там са въведени и съвременните названия на тези линии)[1].

Figure 4: The first page of the Team's paper.

6.3 On the Third Floor of the IET

The final educational goal of the experimental teaching was to foster a kind of synthetic competence. This means to examine the students' knowledge-skills-attitude (KSA) package for multifunktionality and transferability. The didactical innovation was in the team-competence we developed. The stimulus of the Team to continue studying the conics was the presentation of the project at a school symposium and then at an international conference for school students.

6.3.1 Implementation of Old-tech

The pre-computer technology for visualization of envelopes require ingenuity and deep understanding of the matter. The Team prepared a short movie about paper-and-pencil drawing of a parabolic envelope (Figure 5).

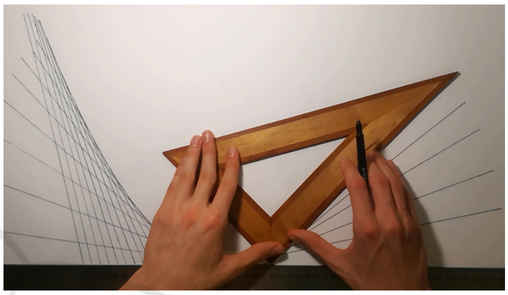


Figure 5: An old-tech method performed and filmed by the Team.

The theoretical base for this drawing relates to another parametrization of the tangents to a parabola: the right-angled vertex of the wooden triangle moves along the line, which is located in the middle between the focus and the directrix. The Team proved the corresponding theorem relying on the parabola construction algorithm.

6.3.2 Explanation of the Terminology

The definitions for conics we adopted do not explain the terminology 'conic sections', but we did not need other definitions for the project. However, it was good manner for the Team to show their classmates the origin of the concept.

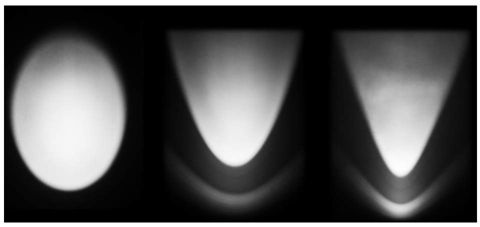


Figure 6: The Team's illustration of conic sections.

For us, it was another opportunity to check the transferability and multifunktionality of the Team's

KSA-package. Therefore, we urge the Team to make the images on Figure 6 and to argue that the three curves, which appears on the wall, are sections of conic shape by plane.

6.3.3 Searching for Applications

Our role on the final stage of the experimental teaching was to state some guidelines for informal learning in accordance with the findings of Petrovic (2018). The didactical goal we stated to the Team was to find appropriate implementations of the reflective properties of the conics. We adopted a kind of tutoring style to force the multifunktionality of the Team's KSA-package beyond the context of its development. The testing area for the transferability of the students' knowledge and skills was real-life applications of conic surfaces.

We already discussed some potential applications of hyperbolic surfaces in the Scholium above. The implementation of the reflective property of parabolic surface in modern communications could be observed just walking around the city (Figure 7, left).

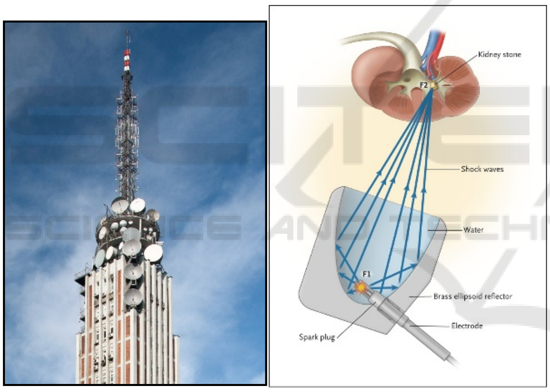


Figure 7: Applications of conic sections surfaces.

More sophisticated was the case with the ellipse. To find some proper applications, the Team was urged to investigate different areas including professional medical sources. The image in the Figure 7 (right) illustrates how an elliptic surface focuses shockwaves for pulverizing renal and ureteric stones (Pearl, 2012). The Team prepared an explanation of the theoretical background of how the lithotripter works.

7 CONCLUSIONS

On the top floor of the IET, the Team demonstrates a large arsenal of multifunctional knowledge and skills cemented by a positive attitude toward mathematics and ICT. The collaboration between the students allowed reaching the stated educational goals via a synergetic effect, which was a kind of revelation for us.

The team synthetic competence developed on this final stage of the experimental teaching encouraged us to continue our individual experimental teaching with the same students for the upcoming school term. The complex way we introduced conics created a solid fundament for consideration more advanced mathematical topics. We plan to enlarge students' synthetic competence entering higher mathematics via computer algebra system.

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Authors' contributions: D. Dimitrov – management and didactical support; B. Lazarov – conceptual frame, tutoring and the paper text.

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APPENDIX

We follow the notations from section 4.

***Proof of Lemma* 1.** Consider the configuration on Figure 8. We have $P \in s_{F_2K} \Longrightarrow F_2P = KP$. Now

$$F_1P - F_2P = (F_1K + KP) - F_2P =$$
$$(F_1K + KP) - KP = F_1K = c \Longrightarrow P \in \mathrm{H}. \square$$

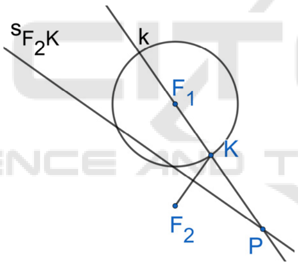

Figure 8: Configuration of Lemma 1.

***Proof of Lemma* 2.** Suppose the contrary: s_{F_2K} has a second common point Q with H, i.e. there exists $\triangle QF_1K$. Let $Z = F_1Q \cap k$ (Figure 9). Now:

$$Q \in \mathrm{H} \Longrightarrow QF_1 - QF_2 = c \Longrightarrow$$
$$(QZ + ZF_1) - QF_2 = c \Longrightarrow$$
$$(QZ + c) - QF_2 = c \Longrightarrow QZ = QF_2.$$
$$Q \in s_{F_2K} \Longrightarrow QF_2 = QK \Longrightarrow QZ = QK.$$

Applying the triangle inequality for $\triangle QF_1K$, we get

$$QK + KF_1 > QF_1 = QZ + ZF_1 \quad \Longrightarrow KF_1 > ZF_1,$$

which is a contradiction. $\square$

***Proof of Theorem* 1.** According to Lemma 2, s_{F_2K} is tangent to H for every $K \in k_1 \cup k_2$. Now let P be an arbitrary point at H such that $PF_1 - PF_2 = c$. Let $K = F_1P \cap k$. Hence $PK = PF_1 - c = PF_2$, i.e. $P \in s_{F_2K}$, which is tangent to H. $\square$

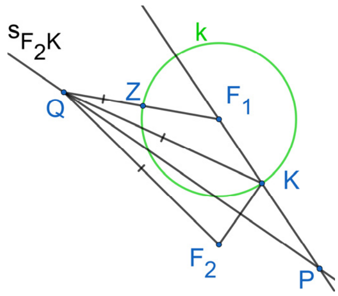

Figure 9: Configuration of Lemma 2.

***Proof of Theorem* 2.** Consider the configuration in Figure 10 where A is an arbitrary point inside the hyperbola branch containing F_2. Let AF_1 meets this branch at P and $AF_1 \cap k = K$. Denote by M the midpoint of KF_2. We have to prove that the rays $AP^{\rightarrow}$ and $PF_2^{\rightarrow}$ conclude equal angles with the tangent to H at P. According to Lemma 2, this tangent is s_{F_2K}. Following the notations in Figure 10, we have to prove that $\alpha = \beta$.

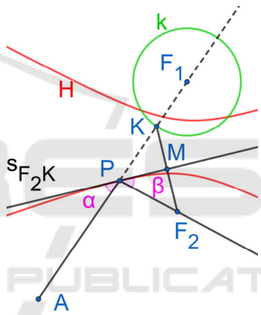

Figure 10: Configuration of Theorem 2.

Since $P \in s_{F_2K}$, then $KP = PF_2$. In the isosceles triangle KPF_2, the segment bisector of KF_2 is angle bisector of $\angle KPF_2$. Hence, $\beta = \angle KPM$. Furthermore, $\alpha = \angle KPM$ as vertical angles. Thus $\alpha = \beta$. $\square$