Implementation of Centralized MPC on the Quadruple-tank Process with Guaranteeing Stability

Roza Ranjbar[1], Lucien Etienne[1][a], Eric Duviella[1][b] and José María Maestre[2][c]
[1]Institute Mines Telecom Lille Douai, Univ. Lille, F-59000, Lille, France
[2]Dept. Ingeniería de Sistemas y Automática, Universidad de Sevilla, Sevilla, Spain

Keywords: Model Predictive Control, Centralized Control, Control Benchmark, Stability.

Abstract: This work presents an implementation of a stabilizing model predictive control applied to a nonlinear system. In this work, the quadruple-tank system has been considered. For this process, a precise control benchmark was available and worked on previously. To ensure the asymptotic stability of this nonlinear system, we made a discretized linearized model and applied a centralized MPC controller with terminal cost constraint. The effectiveness of the proposed strategy is illustrated by simulations.

1 INTRODUCTION

Optimal control design of systems subject to constraint is an important problem of control theory. A powerful way of investigating this problem is to use model predictive controllers (MPCs) which are known to be popular in many fields of applications (Qin and Badgwell, 2003). MPC uses a model of the system dynamics for computing an optimal control action sequence therefore enhancing the computational requirements while achieving optimal performance (Dua et al., 2006). It solves an open-loop constrained optimization problem at each time step, then it executes only the first control of this sequence. The same procedure is repeated at next time steps (Seung Cheol Jeong and PooGyeon Park, 2005).

One of the major benefits of MPC over the other controllers is that it can manage constraints on states, inputs, and outputs. Thus, it allows a system to operate closer to boundaries (Huang et al., 2017). In addition, MPC has the ability of tracking a consistent sequence of set points at the same time that it guarantees that the constraints are satisfied at all times (Alvarado et al., 2011).

MPC strategies have been considered for linear and nonlinear systems, under a variety of communication schemes such as centralized MPC, decentralized MPC, distributed MPC (Segovia et al., 2019; Fele et al., 2017). In this paper, we propose a framework for analyzing the implementation of a classical centralized MPC to ensure the stability of a popular benchmark example of the quadruple-tank process with nonlinear dynamics (Johansson, 2000). We will also emphasize that an optimally controlled system is not necessarily stable and the stability is not ascertained by the use of a finite horizon optimal controller (Kalman et al., 1960; Pannocchia, 2012; Scokaert and Rawlings, 1998).

Related works: Previous researches have been done to provide sufficient conditions for the stability of a MPC controller. Since (Mayne et al., 2000) indicates that stability is an overriding necessity resulting in varied proposals for a MPC and its formulations. Later on, (Cueli and Bordons, 2008) studied a case (both constrained and unconstrained) for deriving the stability criterion that could be ensured under some specific assumptions. Afterwards, (Maiworm et al., 2015), by using a scenario tree, proved how to ensure a reasonable level of stability in the performance of the MPCs.

Contributions: Based on the approach of (Scokaert and Rawlings, 1998) and by using continuity arguments, the main contribution of this paper is to provide sufficient conditions for the stability of a nonlinear system comprised of four-tanks (as a representative of a water network) controlled with a centralized MPC. Namely, we propose a framework for proving asymptotic stability of a Lipschitz nonlinear system using a discretized linearized model for the MPC controller synthesis. Then we apply this result

[a] https://orcid.org/0000-0003-0931-843X
[b] https://orcid.org/0000-0002-1622-0994
[c] https://orcid.org/0000-0002-6343-5445

Ranjbar, R., Etienne, L., Duviella, E. and Maestre, J.
Implementation of Centralized MPC on the Quadruple-tank Process with Guaranteeing Stability.
DOI: 10.5220/0009827700560062
In *Proceedings of the 17th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2020)*, pages 56-62
ISBN: 978-989-758-442-8

on the four-tank benchmark example. The proposed stability outcomes can be applied to any other system which satisfies the assumptions made in section 3.

Outline: Section 2 introduces the the dynamical system under consideration as well as the control scheme. In Section 3, the stability analysis is conducted. Section 4 present the benchmark example. The results are discussed in Section 5 while in Section 6 some concluding remarks are presented.

2 PROBLEM STATEMENT

2.1 System Dynamics

Consider a nonlinear system

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x,u), \tag{1}$$

with $(x,u) = (0,0)$ an equilibrium point such that f is locally Lipschitz : $\forall u \in \mathcal{U}, x_1, x_2 \in \mathcal{X}$

$$||\mathcal{A}x_1 + f(x_1,u) - \mathcal{A}x_2 - f(x_2,u))|| \leq L||x_1 - x_2||,$$

where $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{D} \subset \mathbb{R}^n$ are open set containing 0 with non empty interior.

We make the following assumptions

- A1: $||f(x,u)|| \leq \gamma(|x|)||x||$ with $\gamma(|x|)$ a class $\mathcal{K}$ function,
- A2: $(\mathcal{A},\mathcal{B})$ is stabilizable.

First we treat the nonlinear term $f(x,u)$ as a perturbation/noise. Close to equilibrium assumption A1 quantifies the fact that the nonlinear term can be neglected. We have

$$\dot{x} \approx \mathcal{A}x + \mathcal{B}u. \tag{2}$$

Given $h > 0$ a sampling period, we will introduce the exact discretization of $(\mathcal{A},\mathcal{B})$ by (A,B). Writing $x_k = x(kh)$ and $u_k = u(kh)$, we obtain a general nth-order discrete-time linear state-space description which takes the following form

$$x_{k+1} = Ax_k + Bu_k, \tag{3}$$

where $x_k \in \mathbb{R}^n$. Assigning the x_0 as the initial condition and $u_k \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ as the discrete time input, $x(k) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is system's state. $\mathcal{U}$ is the set of admissible input and we assume it has a non empty interior, while $\mathcal{X}$ is the set of admissible state. We also assume that it has a non empty interior (Rawlings and Mayne, 2009).

2.2 Model Predictive Controller

By applying a centralized MPC for the system (3) the following optimization problem should be solved

$$\min_{\{u_{i|k}\}_{i=k}^{k+H_p-1}, \{x_{i|k}\}_{i=k}^{k+H_p}} J(\{u_{i|k}\}_{i=k}^{k+H_p-1}, \{x_{i|k}\}_{i=k}^{k+H_p}), \tag{4}$$

with $\{u_{i|k}\}_{i=k}^{k+H_p-1} \triangleq \{u_{k|k}, u_{k+1|k}, \ldots, u_{k+H_p-1|k}\}$ and $\{x_{i|k}\}_{i=k}^{k+H_p-1} \triangleq \{x_{k|k}, x_{k+1|k}, \ldots, x_{k+H_p-1|k}\}$, with

$$J(\{u_{i|k}\}_{i=k}^{k+H_p-1}, \{x_{i|k}\}_{i=k}^{k+H_p}) = \sum_{i=k}^{k+H_p-1} x_{i|k}^T Q x_{i|k} + u_{i|k}^T R u_{i|k} + x_{k+H_p|k}^T Q_f x_{k+H_p|k},$$

Here Q is the weight associated to the states, R the weight associated to the outputs, Q_f is a terminal cost on the state and H_p is the prediction horizon. Q_f is chosen to be the solution to the discrete time Riccati equation associated with (A,B,Q,R), i.e. the solution to

$$Q_f = A^T Q_f A - (A^T Q_f B)(R + B^T Q_f B)^{-1}(B^T Q_f A) + Q$$

subject to the following constraints

$$x_{i+1|k} = Ax_{i|k} + Bu_{i|k}, i \in \{k, \ldots, k+H_p-1\}, \tag{5a}$$

$$u_{i|k} \in \mathcal{U}, i \in \{k, \ldots, k+H_p-1\}, \tag{5b}$$

$$x_{j|k} \in \mathcal{X}, j \in \{k, \ldots, k+H_p\}, \tag{5c}$$

$$x_{k|k} = x_k. \tag{5d}$$

Constraint (5a) shows the state equation presented in (5); (5b) describes the feasible inputs and (5c) the feasible states. Finally, the constraint (5d) represents the system's initial condition. In this centralized approach, only the first input $u_{k|k}$ is applied to the system (see (6)) and the others are being neglected according to the receding-horizon philosophy (Richter et al., 2009) (which the control is repeated in this philosophy at every time-step and gives the information of the new state). The following action will be implemented at each time step

$$u_k^{MPC} \triangleq u_{k|k}. \tag{6}$$

3 STABILITY ANALYSIS

Let us recall the dynamics of the system under consideration

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x,u).$$

First assume that $x_0 \in \mathcal{X}_0 \subset \mathcal{X}$, where $\mathcal{X}_0$ is the set of points in $\mathcal{X}$ such that the solution of the (discrete time) optimization problem is given by

$$u_0^{MPC} = -(B^T Q_f A)x_0, \forall k \in \mathcal{N}.$$

We note Q_f the quadratic Lyapunov function associated with this $K := (B^T Q_f A)$. We define:

$$\mathcal{P}_c = \{x \in \mathcal{X}_0 \,|\, x^T Q_f x < c\}. \quad (7)$$

This set always exist and is not empty.

Lemma 1. *there exists a $\underline{h} > 0$ such that solving the optimization problem* (4) *with any discretization time $h < \underline{h}$ and without constraints leads to a controller K, and the solution to the associated solution to the Riccati equation Q_f such that $(\mathcal{A} - \mathcal{B}K)'Q_f + Q_f(\mathcal{A} - \mathcal{B}K) < -\varepsilon I$ for some $\varepsilon > 0$.*

Proof. This follows from (Kailath, 1980) Ch2 sec. 2.6. (see appendix). □

Lemma 2. *Considering the dynamical system* (1) *with Lipschitz constant L on the set X, and $e(t) = x(t) - x_0$ for $t \in [0, h[$ one has:*

$$|e(t)| \leq \frac{Lh}{1 - Lh}|x(t)|.$$

Proof. The proof follows the same line as the one developed in (Tabuada, 2007) *Event-triggered real-time scheduling of stabilizing control tasks* Theorem III.1. (see appendix). □

We define the set $\mathcal{S}_{\frac{\varepsilon}{4|Q_f|}} = \{x \in \mathcal{X}_0 | \gamma(|x|) \leq \frac{\varepsilon}{4|Q_f|}\}$. We define the set

$$\mathcal{P}_* = \max_c \mathcal{P}_c \subset (\mathcal{X}_0 \cap \mathcal{S}_{\frac{\varepsilon}{4|Q_f|}}). \quad (8)$$

Such a set exist and has non empty interior. Note that since $\mathcal{X}_0$ and $\mathcal{S}_{\frac{\varepsilon}{4|Q_f|}}$ contains an open neighborhood of the origin so does $\mathcal{P}_*$.

Theorem 1. *For $h < \frac{\varepsilon}{L(8|Q_f BK| + \varepsilon)}$, $x_0 \in \mathcal{P}_*$ system* (1) *is locally exponentially stable under the model predictive control policy defined in* (6).

Proof. We consider the $e(t) = x(t) - x_0$ for $t \in [0, h[$

Considering a Lyapunov function $V(x) = x'Q_f x$ $x_0 \in \mathcal{P}_*$

We have

$$\dot{V}(x) = \dot{x}'Q_f x + x'Q_f \dot{x},$$

$$\dot{V}(x) = \big(Ax + Bu + f(x,u)\big)'Q_f x + x'Q_f\big(Ax + Bu + f(x,u)\big)$$

Considering first $t \in [0, h[$, given the fact that $u = -Kx_0 = -K(x - e)$ (with $K = (B^T Q_f A)$)

$$\begin{aligned}\dot{V}(x) = & x'(A_{cl}'Q_f + Q_f A_{cl})x \\ & + e'D'Q_f x + x'Q_f De \\ & + f(x, K(e - x))'Q_f x + x'Q_f f(x, K(e - x)),\end{aligned}$$

with $A_{cl} = A - BK, D = BK$.

From Lemma 1

$$\dot{V}(x) \leq -\varepsilon|x|^2 + 2||Q_f D|||x||e| + 2||Q_f|||x||f(x, K(e - x))|,$$

from A_1 and the definition of $\mathcal{P}_*$

$$|f(x, K(e - x))| \leq \frac{\varepsilon}{4|Q_f|}|x|,$$

from Lemma 2 and the definition of h

$$|e| \leq \frac{\varepsilon}{8|Q_f D|}|x|.$$

Therefore for $t \in [0, h[$

$$\dot{V}(x) \leq -\frac{\varepsilon}{2}|x|^2.$$

So it follows by definition of $\mathcal{P}_*$ that at all time $t \in [0, h[$ $x(t) \in \mathcal{P}_*$. Since x is continuous $V(x)$ is also continuous and $x(h^+) \in \mathcal{P}_*$. for all $t \in [h, 2 * h[$ The control gain is given by $u(t) = -Kx(h)$ and we can use the previous argumentation iteratively. Defining $lambda_{\max}(Q_f)$ (resp. $lambda_{\min}(Q_f)$) the biggest (resp. smallest) eigenvalue of $lambda_{\max}(Q_f)$ We have $V(x) \leq \lambda_{\max}(Q_f)|x|^2$. One has by integrating

$$\dot{V}(x(t)) \leq -\frac{\varepsilon}{2\lambda_{\max}(Q_f)}V(x(t))$$

So

$$|x(t)| \leq e^{-\frac{\varepsilon t}{4\lambda_{\max}(Q_f)}}\sqrt{\frac{\lambda_{\max}(Q_f)}{\lambda_{\min}(Q_f)}}|x_0|.$$

One conclude that the original system is locally (i.e. when $x_0 \in \mathcal{P}_*$) exponentially stable. □

4 DESCRIPTION OF THE BENCHMARK

A schematic diagram of the benchmark is shown in Figure 1 (Alvarado et al., 2011). The objective of the process is to control the water levels (h_1, h_2) in the lower tanks using two pumps.

The inputs of this process are pumps' flows (q_a, q_b) and the outputs come from measuring the water level in tanks (h_i).

This model is identified by the following differential equations

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}q_a, \\ \frac{dh_2}{dt} &= -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_a}{S}q_b, \\ \frac{dh_3}{dt} &= -\frac{a_3}{S}\sqrt{2gh_3} + \frac{1 - \gamma_b}{S}q_b, \\ \frac{dh_4}{dt} &= -\frac{a_4}{S}\sqrt{2gh_4} + \frac{1 - \gamma_a}{S}q_a,\end{aligned} \quad (9)$$

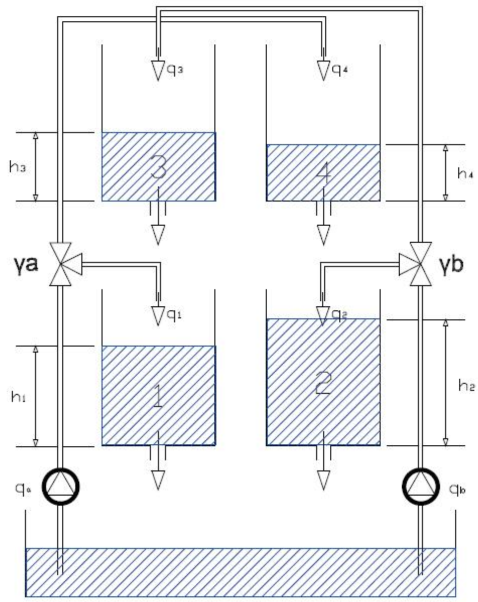

Figure 1: Johansson's quadruple-tank process diagram.

where $S(m^2)$ is the cross-section of all four-tanks, $h_i(m)$ and $a_i(m^2)$, $i \in \{1,2,3,4\}$ mention the water level and the discharge constant of tank *i*, respectively. A voltage is applied to pump *j* which provides the discharge q_j (m^3h^{-1}) with the corresponding ratio of γ_j and $j \in \{a,b\}$. $g(ms^{-2})$ is denoted as the gravitational acceleration. The parameter values are estimated experimentally in the laboratory and are presented in Table1 (Alvarado et al., 2011).

Upon the four-tank process which has been illustrated above as a benchmark, the centralized controller (MPC) is tested, in order to analyze its proficiency. The model and controller is implemented in SIMULINK and the MPC controller is computed using CVX (Boyd and Vandenberghe, 2004).

4.1 Prediction Model and Simulation

For applying the centralized MPC, there should be a linear prediction model which is obtained through linearizing (9) around an equilibrium point (h^0,q^0). The operating point is assigned from the equilibrium levels presented in the previously referenced table. Consider the variables around the operating points as follow

$$x_i = h_i - h_i^0 \text{ , } i \in 1,2,3,4$$
$$u_1 = q_a - q_a^0,$$
$$u_2 = q_b - q_b^0.$$

Table 1: Parameters of the quadruple-tank.

Parameters	Value	Unit	Description
h_{1max}	1.36	m	Maximum level of the tank 1
h_{2max}	1.36	m	Maximum level of the tank 2
h_{3max}	1.30	m	Maximum level of the tank 3
h_{4max}	1.30	m	Maximum level of the tank 4
h_{min}	0.2	m	Minimum level in all cases
q_{amax}	3.26	m^3/h	Maximum flow of q_a
q_{bmax}	4	m^3/h	Maximum flow of q_b
q_{min}	0	m^3/h	Minimum flow of q_a and q_b
a_1	$1.31e-4$	m^2	discharge constant of tank 1
a_2	$1.51e-4$	m^2	discharge constant of tank 2
a_3	$9.27e-5$	m^2	discharge constant of tank 3
a_4	$8.82e-5$	m^2	discharge constant of tank 4
S	0.06	m^2	Cross-section of the tanks
γ_a	0.3		Parameter of the 3-way valve
γ_b	0.4		Parameter of the 3-way valve
h_1^0	0.65	m	Linearization level of tank 1
h_2^0	0.66	m	Linearization level of tank 2
h_3^0	0.65	m	Linearization level of tank 3
h_4^0	0.66	m	Linearization level of tank 4
q_a^0	1.63	m^3/h	Linearization flow of q_a
q_b^0	2.00	m^3/h	Linearization flow of q_b

The linearized continuous-time state-space model becomes as

$$\frac{dx}{dt} = \mathcal{A}_c x + \mathcal{B}_c u, \tag{10}$$

in which, $x = [x_1,x_2,x_3,x_4]$, $u = [u_1,u_2]$, $Q = C_c^T C_c$, $R = I$ and the matrices are as follow

$$\mathcal{A}_c = \begin{bmatrix} \frac{-1}{\tau 1} & 0 & \frac{1}{\tau 3} & 0 \\ 0 & \frac{-1}{\tau 2} & 0 & \frac{1}{\tau 4} \\ 0 & 0 & \frac{-1}{\tau 3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau 4} \end{bmatrix}, \mathcal{B}_c = \begin{bmatrix} \frac{\gamma_a}{S} & 0 \\ 0 & \frac{\gamma_b}{S} \\ 0 & \frac{1-\gamma_b}{S} \\ \frac{1-\gamma_a}{S} & 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

with $\tau = \frac{S}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$ and $i \in \{1,2,3,4\}$. For the implementation of the centralized MPC, equation (10) is discretized with a sampling time of Ts=5 seconds. We have chosen the prediction horizon as Hp=20 seconds.

4.2 Control Objectives

There are some issues which should be taken into consideration before using the controller.

- Modelling: the class of a model is highly dependant on the type of the controller which is going to be used and also the aim of the controlling item. For example, it is an important decision to use a linear or nonlinear model.
- Targets: there are a variety of attributes assigned by the employed specific type of the controller. These attributes come as optimality, stability, feasibility, *etc*.

- Required joint software: optimization routines, simulation routines, *etc*.

5 RESULTS

The main purpose of the benchmark is keeping the water levels of tanks 1 and 2 as close as possible to their referenced levels. Thus, different reference shifts are inserted in the benchmark used in (Alvarado et al., 2011) to examine different equilibrium points. As shown in Figures 2 and 3, the reference signals change every 1000 seconds and the initial values are according to the previously quoted operating points in the linearization of (10).

Figure 2(a) displays the water level and the steady state in tank 1. At the time 2000s, there is a difference of around 0.05m in the mentioned levels. Also, at the time 3000s there is a similar deviation of 0.05m. Figure 2(b) displays the water level and the steady state in tank 2. In comparison to tank 1, The convergence seems faster however some steady state error is present (unlike tank 1 where there is no steady state error). At the time 2000s, there is a difference of around 0.03m in the mentioned levels and at the time 3000s, this amount decreases up to 0.02m.

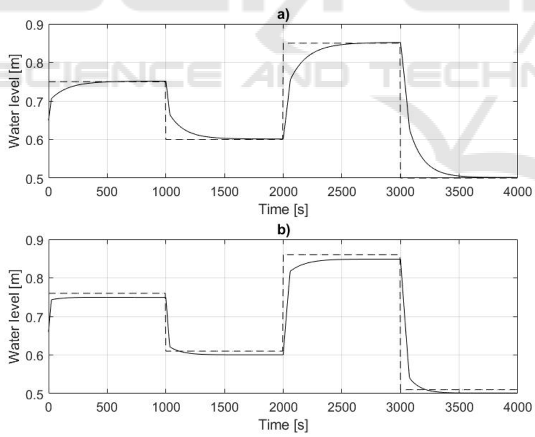

Figure 2: Water levels in tanks 1 (fig. a)) and 2 (fig. b)).

Figure 3, related to the tanks 3 and 4, reveals some overshoots in water levels. For instance at around the time of 2000s, tank 3 shows a sharp rise of the water level reaches to 0.90m while the steady state at this time is 0.85m. This tank displays a deep fall at around the time 3000s and reaches the water level of 0.44m when the steady state at this time is 0.50m.
At around the time of 2000s, tank 4 shows a sharp rise of the water level reaches to 0.91m while the steady state at this time is 0.85m. This tank displays a deep fall at around the time 3000s and reaches the water level of 0.46m when the steady state at this time is 0.51m. These overshoots are probably occurring due to the points' far distance from the linearization points.

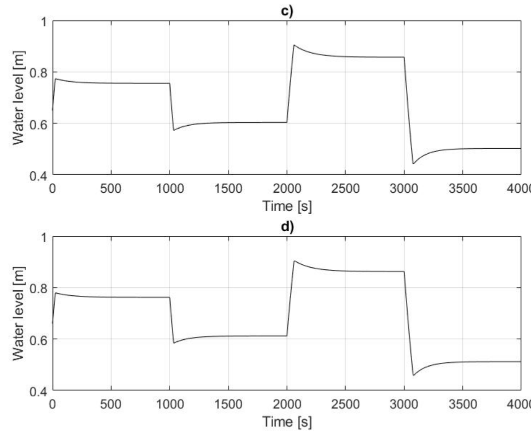

Figure 3: Water levels in tanks 3 (fig. a)) and 4 (fig. a)).

Moreover, the controller's values are depicted in Figure 3. Fig. 3(a) represents the first control in which the maximum allowable flow is $q_{amax} = 3.26m^3/h$. Fig. 3(b) shows the second control in which the maximum allowable flow is $q_{bmax} = 4m^3/h$. In overall, the saturation appears to be well handled by the proposed MPC.
Therefore, the controller is generally able to deal with the operational constraints. Although, the nonlinear system was supposed to be locally exponentially stable, there were some inaccuracies possibly due to the numerical errors or defining the sampling time step in the simulation step.

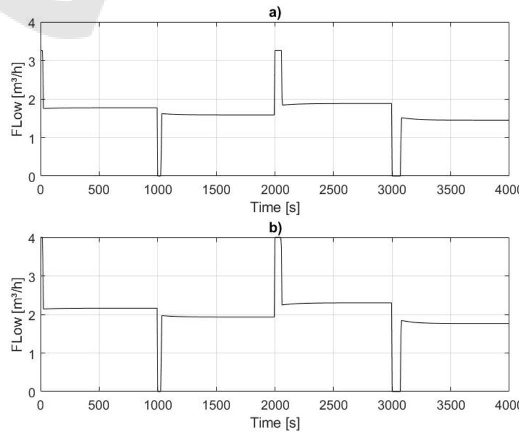

Figure 4: The controller's values q_a in fig. a) and q_b in fig. b).

6 CONCLUSION

In this work, the problem of MPC control design for a four-tank benchmark model was considered. First, it was shown that under suitable constraints the nonlinear continuous time system can be stabilized by a linear discrete time controller. Moreover, we have conducted simulations that show the good performance of the control algorithm.

Further work will provide tighter estimates of the region where stability and feasibility can be guaranteed. Another promising research direction is the use of hybrid/multi model in order to enhance the control performance and robustness.

ACKNOWLEDGEMENTS

The authors would like to thank the Regional Council Hautes-de-France for its support.

REFERENCES

Alvarado, I., Limon, D., de la Peña, D. M., Maestre, J., Ridao, M., Scheu, H., Marquardt, W., Negenborn, R., Schutter, B. D., Valencia, F., and Espinosa, J. (2011). A comparative analysis of distributed mpc techniques applied to the hd-mpc four-tank benchmark. *Journal of Process Control*, 5:800–815.

Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Cueli, J. R. and Bordons, C. (2008). Iterative nonlinear model predictive control. stability, robustness and applications. *Control Engineering Practice*, 16(9):1023 – 1034.

Dua, P., Doyle, F. J., and Pistikopoulos, E. N. (2006). Model-based blood glucose control for type 1 diabetes via parametric programming. *IEEE transactions on bio-medical engineering*, 53:1478–1491.

Fele, F., Maestre, J. M., and Camacho, E. F. (2017). Coalitional control: Cooperative game theory and control. *IEEE Control Systems Magazine*, 37(1).

Huang, Y., Wang, H., Khajepour, A., He, H., and Ji, J. (2017). Model predictive control power management strategies for hevs: A review. *Journal of Power Sources*, 341:91–106.

Johansson, K. H. (2000). The quadruple-tank process: a multivariable laboratory process with an adjustable zero. *IEEE Trans. Contr. Sys. Techn.*, 8:456–465.

Kailath, T. (1980). *Linear systems*, volume 156. Prentice-Hall Englewood Cliffs, NJ.

Kalman, R. E. et al. (1960). Contributions to the theory of optimal control. *Bol. soc. mat. mexicana*, 5(2):102–119.

Maiworm, M., Bäthge, T., and Findeisen, R. (2015). Scenario-based model predictive control: Recursive feasibility and stability. *IFAC-PapersOnLine*, 48(8):50 – 56. 9th IFAC Symposium on Advanced Control of Chemical Processes ADCHEM 2015.

Mayne, D., Rawlings, J., Rao, C., and Scokaert, P. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789 – 814.

Pannocchia, G. (2012). Course on model predictive control. *bio engineering and robotics research center*, pages 1–33.

Qin, S. J. and Badgwell, T. A. (2003). A survey of industrial model predictive control technology. *Control engineering practice*, 11(7):733–764.

Rawlings, J. B. and Mayne, D. Q. (2009). Model predictive control: Theory and design, nob hill pub. *Madison, Wisconsin*.

Richter, S., Jones, C. N., and Morari, M. (2009). Real-time input-constrained mpc using fast gradient methods. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, pages 7387–7393. IEEE.

Scokaert, P. O. and Rawlings, J. B. (1998). Constrained linear quadratic regulation. *IEEE Transactions on automatic control*, 43(8):1163–1169.

Segovia, P., Rajaoarisoa, L., Nejjari, F., Duviella, E., and Puig, V. (2019). A communication-based distributed model predictive control approach for large-scale systems. In *58th IEEE Conference on Decision and Control (CDC)*. IEEE.

Seung Cheol Jeong and PooGyeon Park (2005). Constrained mpc algorithm for uncertain time-varying systems with state-delay. *IEEE Transactions on Automatic Control*, 50(2):257–263.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685.

APPENDIX

Proof of Lemma 1

Proof. Solving the optimization problem (4) with any discretization time $h < \underline{h}$ leads to an LQR controller K associated with a Cost $x_0' Q_f x_0$ Such that $(A-BK)'Q_f(A-BK) - Q_f = -Q$.

Since $A = I + h\mathcal{A} + o(h), B = h\mathcal{B} + o(h)$, defining $W(h) := I + h(\mathcal{A} - \mathcal{B}K)$

$$(W(h) + o(h))'Q_f(W(h) + o(h)) - Q_f < -Q/2$$

$$h(\mathcal{A} - \mathcal{B}K + o(h))'Q_f + hQ_f(\mathcal{A} - \mathcal{B}K) + o(h) < -Q/2,$$

Therefore there exist h small enough such that

$$(\mathcal{A} - \mathcal{B}K)'Q_f + Q_f(\mathcal{A} - \mathcal{B}K) < -S,$$

With S a symmetric positive definite matrix therefore

$$(\mathcal{A}-\mathcal{B}K)'Q_f + Q_f(\mathcal{A}-\mathcal{B}K) < -\lambda_{\min}(S)I,$$

□

Proof of Lemma 2

Proof. For the dynamical system (1) with Lipschitz constant L and considering the dynamics of $\frac{|e|}{|x|}$ one has

$$\frac{d}{dt}\frac{|e|}{|x|} = \frac{d}{dt}\frac{(e^Te)^{1/2}}{(x^Tx)^{1/2}},$$

$$\frac{d}{dt}\frac{|e|}{|x|} = \frac{(e^Te)^{-1/2}e^T\dot{e}(x^Tx)^{1/2} - (x^Tx)^{-1/2})x^T\dot{x}(e^Te)^{1/2}}{(x^Tx)^{1/2}},$$

$$\frac{d}{dt}\frac{|e|}{|x|} = \frac{e^T\dot{x}}{|e||x|} - \frac{x^T\dot{x}}{|x||x|}\frac{|e|}{|x|},$$

$$\frac{d}{dt}\frac{|e|}{|x|} \leq \frac{|e||\dot{x}|}{|e||x|} - \frac{|x||\dot{x}|}{|x||x|}\frac{|e|}{|x|},$$

$$\frac{d}{dt}\frac{|e|}{|x|} \leq \left(1+\frac{|e|}{|x|}\right)\frac{|\dot{x}|}{|x|},$$

$$\frac{d}{dt}\frac{|e|}{|x|} \leq \left(1+\frac{|e|}{|x|}\right)\frac{L(|x|+|e|)}{|x|},$$

$$\frac{d}{dt}\frac{|e|}{|x|} \leq L\left(1+\frac{|e|}{|x|}\right)^2$$

integrating from time 0 to h the equation $\dot{\phi} = L(1+\phi^2)$ with $\phi(0) = 0$ one obtains

$$|e(t)| \leq \frac{Lh}{1-Lh}|x(t)|.$$

□