Robustness Estimation of Large Deviations in Linear Discrete-time Systems with Control Signal Delay

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Abstract: The article deals with robustness estimation of large deviations in free motion of linear discrete-time systems to parameter variations of the state matrix. A tracking discrete-time system with the modal control law is considered in the paper. The modal control law is designed taking into account the value of delay and the deviation. It is assumed that parameters of the system are linearly dependent on the uncertainties. The problem is solved with the state space approach and the sensitivity theory methods. An upper bound estimation of trajectory deviations for discrete-time systems is obtained. The estimation contains the condition number of the eigenvectors matrix of the system state matrix. Therefore, sensitivity functions of singular values of the eigenvectors matrix are used to calculate the robustness estimation of the deviations. Based on the obtained equations, an algorithm for the robustness estimation of large deviations in linear discrete-time systems with parametric uncertainties is proposed. Two cases of control signal delay are considered in the paper. The first case relates to predictable delay of control signal, and the second one relates to unpredictable delay of control signal. The results are supported with an examples.

1 INTRODUCTION

Large deviations (peak effects) in the free motion of a linear system occur due to nonzero initial conditions in the absence of exogenous input signal. The large deviation problem has been investigated by specialists in control theory and signal processing for a long time. Firstly, the relationship between system poles and large deviations of the motion of the system was discussed by (Feldbaum, 1948) and (Izmailov, 1987). The problem of large deviations in linear systems with observer was considered by (Polotskij, 1981). Also, this problem is presented for switching systems in (Liberzon, 2003; Vunder and Dudarenko, 2018a), and for cascade control systems in (Sussman and Kokotovic, 1991), where the result of R.N. Izmailov was generalized to obtain estimations of the deviations for the outputs. Recent papers (Polyak and Smirnov, 2016; Vunder et al., 2016) continued the study in that field for different values of system poles, and new results for estimation of upper bound of deviations were obtained with the linear matrix inequality in (Polyak and Smirnov, 2016) and with the condition number of eigenvectors matrix in (Vunder and Ushakov, 2017).

Deviations in discrete-time linear systems were considered in the research groups (Shcherbakov, 2017; Vunder and Ushakov, 2015) also. Moreover, approach to estimation of deviations in discrete-time linear systems with 'predictable' and 'unpredictable' control signal delay was improved and proposed in the paper (Dudarenko et al., 2019). Recently, the problem of robustness estimation of deviations in free motion of linear dynamic system was investigated by (Khlebnikov, 2018; Ahiyevich et al., 2018; Vunder and Dudarenko, 2018b). But it is still a challenge for the scientists to find an optimal and universal solution of the large deviation problem.

This paper is extension of previous results of the authors to the case of discrete-time systems with parametric uncertainties. Robustness estimation of large deviations in linear stable discrete-time systems with parametric uncertainties is considered in the paper. A tracking discrete-time systems with the modal control law are discussed, where the modal control law is designed taking into account the value of delay and the deviation. It is assumed that parameters of the system are linearly dependent on uncertainties. The

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problem is solved with the sensitivity analysis (Eslami, 1994) and with the state-space approach. An algorithm for robustness estimation of large deviations in linear discrete-time system with parametric uncertainties is proposed. The algorithm can be used for the model description of a linear stable system in the state-space representation with the constant matrices in arbitrary form.

The results of the paper can be useful for the stabilization problems solution and design of control of uncertain linear plants in conditions of time-delay (Polyak et al., 2015; Abidi and Soo, 2019; Liu et al., 2018; Margun and Furtat, 2016).

The paper is laid out as follows. Firstly, an illustrative example of large deviation in a discrete-time system is represented, and equations for the estimation of the upper bound of the deviation are given. Then, the basic expressions for large deviation assessment are described for two cases of control signal delay in discrete-time systems: with predictable delay of control signal and with unpredictable delay of control signal. Thereafter, the approach to the robustness estimations of large deviations in discrete-time system with parametric uncertainties and the algorithm are proposed. Then, example of a discrete-time system with parametric uncertainties is considered, where robustness of large deviations is assessed with the proposed approach. The paper is finished with some concluding remarks.

2 ASSESSMENT OF THE UPPER BOUND OF LARGE DEVIATIONS

Assume a discrete-time system is given by:

$$\mathbf{x}(k+1) = \bar{\mathbf{F}}\mathbf{x}(k); \mathbf{x}(0). \quad (1)$$

where $\bar{F}x(k)$ is stable state matrix with eigenvalues $\sigma\{\bar{F}\} = \left\{ \begin{array}{c} \bar{\lambda}_i = \arg\left(\det\left(\bar{\lambda}I - \bar{F}\right) = 0\right): \\ \operatorname{Im}\left(\bar{\lambda}_i\right) = 0, \bar{\lambda}_i \neq \bar{\lambda}_j; i, j = \overline{1,n}; i \neq j \end{array} \right\}$.

The solution of equation (1) takes the form

$$\mathbf{x}(k) = \bar{\mathbf{F}}^k \mathbf{x}(0). \quad (2)$$

Equation (2) can be rewritten with the vector and matrix norms in the following form

$$\|\mathbf{x}(k)\| = \left\|\bar{\mathbf{F}}^k \mathbf{x}(0)\right\| \leq \left\|\bar{\mathbf{F}}^k\right\| \|\mathbf{x}(0)\|, \quad (3)$$

where $\|*\|$ is any consistent norm here and elsewhere.

The norm behaviour of free motion of discrete-time system (1) is depicted in Figure 1 for two cases of the state matrix $\bar{\mathbf{F}}$.

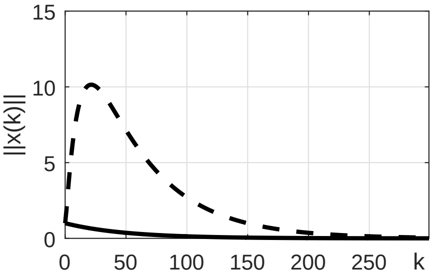

Figure 1: Norm of free motion of a stable discrete-time system for different descriptions of the state matrix $\mathbf{F}_1$ (dashed line) and $\mathbf{F}_2$ (solid line).

There are $\bar{\mathbf{F}}_1 = \begin{bmatrix} 0.9802 & 1.3253 \\ 0 & 0.9139 \end{bmatrix}$ and $\bar{\mathbf{F}}_2 = \begin{bmatrix} 0.9471 & 0.0331 \\ 0.0331 & 0.9471 \end{bmatrix}$. It should be noted, the eigenvalues $\bar{\lambda}_i$ for both cases of $\bar{\mathbf{F}}$ are the same: $\bar{\lambda}_1 = 0.9802$, $\bar{\lambda}_2 = 0.9139$. Obviously, norm behaviour with the matrix $\mathbf{F}_2$ converges to zero monotonically, while norm behaviour with the matrix $\mathbf{F}_1$ has large deviation from the monotonic trajectory. Here, the last case describes large deviation in the system behaviour, that is the researched subject of the paper.

Assessment of the upper bound of large deviations of free motion of discrete-time system (1) is obtained with representation of the state matrix $\bar{\mathbf{F}}$ in the following form

$$\mathbf{M}\bar{\Lambda} = \bar{\mathbf{F}}\mathbf{M}, \quad (4)$$

where $\bar{\Lambda}$ is a diagonal matrix of eigenvalues, $\mathbf{M}$ is a square matrix whose columns are the n linearly independent eigenvectors of $\bar{\mathbf{F}}$. Using (2) and (4), we get

$$\mathbf{M}\bar{\Lambda}^k = \bar{\mathbf{F}}^k \mathbf{M}. \quad (5)$$

Now, combining (2) and (5), we obtain

$$\mathbf{x}(k) = \bar{\mathbf{F}}^k \mathbf{x}(0) = \mathbf{M}\bar{\Lambda}^k \mathbf{M}^{-1}\mathbf{x}(0). \quad (6)$$

Let us form an upper bound of (3)

$$\begin{gathered} \|\mathbf{x}(k)\| = \left\|\mathbf{M}\bar{\Lambda}^k \mathbf{M}^{-1}\mathbf{x}(0)\right\| \leq \\ \|\mathbf{M}\| \left\|\bar{\Lambda}^k\right\| \left\|\mathbf{M}^{-1}\right\| \|\mathbf{x}(0)\| = \\ C\{\mathbf{M}\} \left\| diag\left(\bar{\lambda}_i^k\right); i = \overline{1,n} \right\| \|\mathbf{x}(0)\| \leq \\ C\{\mathbf{M}\} \bar{\lambda}_{\max}^k \|\mathbf{x}(0)\|, \end{gathered} \quad (7)$$

where $C\{\mathbf{M}\} = \|\mathbf{M}\| \left\|\mathbf{M}^{-1}\right\|$ is condition number (Gantmacher, 1990), (Golub and Van Loan, 1996) of the matrix $\mathbf{M}$; $\bar{\lambda}_{\max}$ is a maximum eigenvalue of matrix $\bar{\mathbf{F}}$ that satisfies conditions $Im\left(\bar{\lambda}_{\max}\right) = 0, \bar{\lambda}_{\max} > 0$. Thus, by $\|\mathbf{x}(0)\| = 1$ we have the upper bound:

$$\sup\left(\|\mathbf{x}(k)\|\right)\big|_{\|\mathbf{x}(0)\|=1} = C\{\mathbf{M}\} \bar{\lambda}_{\max}^k. \quad (8)$$

It should be noted, if there is even one pair of close to collinear eigenvectors, then the condition number $C\{\mathbf{M}\}$ can be sufficiently large and, therefore, deviation of free motion of the system become large also.

Main properties of equation (8) are discussed in (Dudarenko et al., 2019).

3 DEVIATIONS IN DISCRETE-TIME SYSTEMS WITH CONTROL SIGNAL DELAY

Any discrete-time control system is a composition of following parts: digital controller, a digital-to-analog converter, analog-to-digital converter and a continuous-time plant. In order to obtain a single mathematical description of this composition, processes are studied at the time instances $t = k \cdot \Delta t$, where k is positive integer, it is called the discrete time; Δt is the sample time. This means that the discrete-time plant is said to be discrete time sampling from continuous-time state and output variables under a piecewise-constant control signal with the duration Δt. Note that a control signal from the digital controller can output both without delay and with delay τ. This fact gives rise to two discrete-time representations of the continuous-time plant.

Consider a linear continuous-time plant:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t); \mathbf{x}(0), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \qquad (9)$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^r, \mathbf{y} \in \mathbb{R}^m$ are state vector, input vector, output vector respectively;$\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$ are state matrix, input matrix, output matrix. If the control of the plant (9) for $t = k \cdot \Delta t$ is realized without delay, then it can be represented as follows:

$$u(t) = u(k), k\Delta t \leq t < (k+1)\Delta t. \qquad (10)$$

Combining (9) and (10) (Zadeh and Desoer, 2008), we get following discrete-time model of the plant:

$$\mathbf{x}(k+1) = \bar{\mathbf{A}}\mathbf{x}(k) + \bar{\mathbf{B}}\mathbf{u}(k); \mathbf{x}(0), \\ \mathbf{y}(k) = \bar{\mathbf{C}}\mathbf{x}(k), \qquad (11)$$

where $k = \arg(t = k\Delta t)$ is discrete time; Δt is sample time; $\dim(\bar{\mathbf{A}}) = \dim(\mathbf{A})$, $\dim(\bar{\mathbf{B}}) = \dim(\mathbf{B})$, $\dim(\bar{\mathbf{C}}) = \dim(\mathbf{C})$, $\bar{\mathbf{A}} = \exp(\mathbf{A}\Delta t)$, $\bar{\mathbf{B}} = (\bar{\mathbf{A}} - I)\mathbf{A}^{-1}\mathbf{B}$, $\bar{\mathbf{C}} = \mathbf{C}$. Analytically, control (10) can be written as:

$$\mathbf{u}(k) = \bar{\mathbf{K}}_g\mathbf{g}(k) - \bar{\mathbf{K}}\mathbf{x}(k), \qquad (12)$$

where $\mathbf{g} \in \mathbb{R}^m$ is an external input; $\bar{\mathbf{K}}_g \in \mathbb{R}^{r\times m}$, $\bar{\mathbf{K}} \in \mathbb{R}^{r\times n}$ are the feed forward matrix and the feedback matrix respectively. Combining (12) and (9), we get discrete-time closed-loop system:

$$\mathbf{x}(k+1) = \bar{\mathbf{F}}\mathbf{x}(k) + \bar{\mathbf{G}}\mathbf{g}(k); \mathbf{x}(0), \\ \mathbf{y}(k) = \bar{\mathbf{C}}\mathbf{x}(k), \\ \varepsilon(k) = \mathbf{g}(k) - \mathbf{y}(k), \qquad (13)$$

where

$$\bar{\mathbf{F}} = \bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}}, \bar{\mathbf{G}} = \bar{\mathbf{B}}\bar{\mathbf{K}}_g, \qquad (14)$$

$\varepsilon(k)$ is a tracking error. Eigenvalues and eigenvectors of the state matrix is given by:

$$\sigma\{\bar{\mathbf{F}}\} = \{\bar{\lambda}_i = \exp(\lambda_i\Delta t) : Im(\bar{\lambda}_i) = 0, \bar{\lambda}_i \neq \bar{\lambda}_j\}, \qquad (15)$$

$$\bar{\mathbf{F}}\xi_i = \bar{\lambda}_i\xi_i; i = \overline{1,n}. \qquad (16)$$

The case of a discrete-time system with the control signal delay is characterized by the increased dimension of the matrices and modification of eigenvector structure of the state matrix. If the control $u(t)$ of the plant (9) for $t = k \cdot \Delta t$ realizes with delay $\tau \leq \Delta t$, then it can be represented as follows (Grigoriev et al., 1983)

$$u(t) = \begin{cases} u(k-1), & k\Delta t \leq t < k\Delta t + \tau; \\ u(k), & k\Delta t + \tau \leq t < (k+1)\Delta t. \end{cases} \qquad (17)$$

Combining (17) and (9), we get following discrete-time model (Grigoriev et al., 1983) of the plant:

$$\mathbf{x}(k+1) = \bar{\mathbf{A}}\mathbf{x}(k) + \bar{\mathbf{B}}_1(\tau)\mathbf{u}(k-1) + \bar{\mathbf{B}}(\tau)\mathbf{u}(k); \\ \mathbf{y}(k) = \bar{\mathbf{C}}\mathbf{x}(k), \qquad (18)$$

where

$$\bar{\mathbf{B}}_1(\tau) = \bar{\mathbf{A}}\left(I - e^{-\mathbf{A}\tau}\right)\mathbf{A}^{-1}\mathbf{B}, \\ \bar{\mathbf{B}}(\tau) = \left(\bar{\mathbf{A}}e^{-\mathbf{A}\tau} - I\right)\mathbf{A}^{-1}\mathbf{B}. \qquad (19)$$

Let us introduce an additional state vector χ, then we get a following discrete-time model

$$\tilde{\mathbf{x}}(k+1) = \begin{bmatrix} \mathbf{x}(k+1) \\ \chi(k+1) \end{bmatrix} = \\ \begin{bmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}}_1(\tau) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \chi(k) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}(\tau) \\ I \end{bmatrix} \mathbf{u} = \\ \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{u}(k); \mathbf{y}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k), \qquad (20)$$

where $\dim(\chi) = \dim(\mathbf{u}) = r$.

Two cases are considered. The first case is called 'unpredictable delay' (or unaccounted delay). In this case the control is the same like for discrete plant without delay. That means an additional dimension (Z^{-1} is discrete-time operator) does not take into account when the control is designed. The control is given by (12), but on account of the modification of the plant model (20) the discrete-time system takes the form:

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{F}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{G}}\mathbf{g}(k); \mathbf{y}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k), \quad (21)$$

where

$$\tilde{\mathbf{F}}(\tau) = \begin{bmatrix} \bar{\mathbf{A}} - \bar{\mathbf{B}}(\tau)\bar{\mathbf{K}} & \bar{\mathbf{B}}_1(\tau) \\ -\bar{\mathbf{K}} & 0 \end{bmatrix},$$

$$\tilde{\mathbf{G}}(\tau) = \begin{bmatrix} \bar{\mathbf{B}}(\tau)\bar{\mathbf{K}}_g \\ \bar{\mathbf{K}}_g \end{bmatrix}, \tilde{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}} & 0 \end{bmatrix} \quad (22)$$

Block diagram representation of system (21) with control (12) is shown in figure 2.

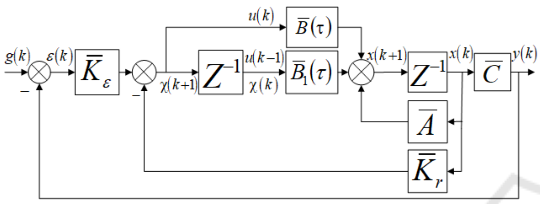

Figure 2: Block diagram of system with unpredictable delay in control.

The upper bound of free motion of system (21) takes the form (8). And it should be noted that a change of condition number $C\{\tilde{\mathbf{M}}\}$ happens even by $\tau = 0$ although eigenvalues set of the matrix $\tilde{\mathbf{F}}(\tau) = \tilde{\mathbf{F}}(0)$ is increased a zero eigenvalue $\tilde{\lambda}_{n+1} = 0$.

The second case is called 'predictable delay' (or accounted delay). In this case the control law takes into account the additional dimension (Z^{-1} discrete-time operator) by an additional feedback $\tilde{\mathbf{K}}_\chi$. The control low for plant (20) takes the form

$$\mathbf{u}(k) = \tilde{\mathbf{K}}_g\mathbf{g}(k) - \tilde{\mathbf{K}}_x\mathbf{x}(k) - \tilde{\mathbf{K}}_\chi\chi(k), \quad (23)$$

Combining (20) and (23), we get the discrete-time closed-loop system (21) with the next matrices

$$\tilde{\mathbf{F}}(\tau) = \begin{bmatrix} \bar{\mathbf{A}} - \bar{\mathbf{B}}(\tau)\tilde{\mathbf{K}}_x & \bar{\mathbf{B}}_1(\tau) - \bar{\mathbf{B}}(\tau)\tilde{\mathbf{K}}_\chi \\ -\tilde{\mathbf{K}}_x & -\tilde{\mathbf{K}}_\chi \end{bmatrix},$$

$$\tilde{\mathbf{G}}(\tau) = \begin{bmatrix} \bar{\mathbf{B}}(\tau)\tilde{\mathbf{K}}_g \\ \tilde{\mathbf{K}}_g \end{bmatrix}, \tilde{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}(\tau) & 0 \end{bmatrix} \quad (24)$$

Control law (23) is formed such that an eigenvalues set of matrix $\tilde{\mathbf{F}}(\tau)$ (24) consists of eigenvalues set of matrix $\bar{\mathbf{F}}$ (22) and an eigenvalue $\tilde{\lambda}_{n+1}$. The eigenvalue $\tilde{\lambda}_{n+1}$ satisfies the condition $\tilde{\lambda}_{n+1} \ll \tilde{\lambda}_i, i = \overline{1,n}$. Block diagram representation of system (21) with matrices (24) and control (23) is shown in figure 3.

The upper bound of free motion of this system satisfies the form (8) as well, where $C\{\tilde{\mathbf{M}}\}$ is condition number of $(n+1)\times(n+1)$ eigenvectors matrix $\tilde{\mathbf{M}}$ of the state matrix $\tilde{\mathbf{F}}$ (24).

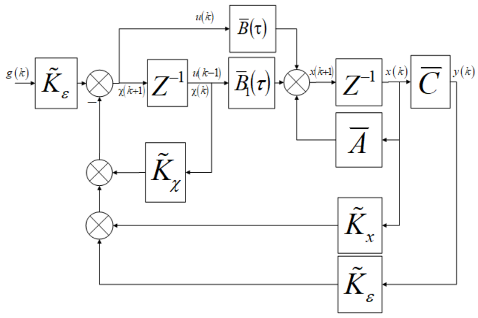

Figure 3: Block diagram of system with predictable delay in control.

4 ROBUSTNESS ESTIMATION

Assume parameters of the state matrix linearly depends on components of the vector of varying stationary parameters $\mathbf{q} = \mathbf{q}_0 + \Delta\mathbf{q}, \mathbf{q} \in \mathbb{R}^p$, where $\mathbf{q}_0$ is vector of nominal value of the parameters $\mathbf{q}$, $\Delta\mathbf{q}$ is vector of variations of the parameters. Then, the state matrix $\bar{\mathbf{F}}(\mathbf{q})$ depends of the vector of varying parameters and system (1) takes the following form:

$$\mathbf{x}(k+1,\mathbf{q}) = \bar{\mathbf{F}}(\mathbf{q})\mathbf{x}(k,q); \mathbf{x}(0) = \mathbf{x}(k,\mathbf{q})|_{k=0}, \quad (25)$$

Therefore, equation (8) can be rewritten as:

$$\|\mathbf{x}(k,\mathbf{q})\| \leq \sup\{\|\mathbf{x}(k,\mathbf{q})\|\} = C\{\mathbf{M}(\mathbf{q})\}\bar{\lambda}^k_{\max}(\mathbf{q})\|\mathbf{x}(0)\|. \quad (26)$$

Consider a ν-sensitivity function $\Theta_\nu(k,\mathbf{q})$ of upper bound (26) to variation of ν component of a vector of parameters $\mathbf{q}_\nu$. The sensitivity function $\Theta_\nu(k,\mathbf{q})$ can be defined as:

$$\Theta_\nu(k,\mathbf{q}) = \left.\frac{\partial(\sup\{\|\mathbf{x}(k,\mathbf{q})\|\})}{\partial q_\nu}\right|_{\mathbf{q}_0} =$$

$$\left(\left.\frac{\partial C\{\mathbf{M}(\mathbf{q})\}}{\partial q_\nu}\right|_{\mathbf{q}_0} \bar{\lambda}^k_{\max} + k\bar{\lambda}^{k-1}_{\max} \left.\frac{\partial\bar{\lambda}_{\max}(\mathbf{q})}{\partial q_\nu}\right|_{\mathbf{q}_0} C\{\mathbf{M}(\mathbf{q})\}\right)\|\mathbf{x}(0)\|. \quad (27)$$

Note, calculation of derivative of conditional number of the modified eigenvectors matrix $C\{\mathbf{M}\} = \|\mathbf{M}\| \cdot \|\mathbf{M}^{-1}\|$ depends on choosing norm. It is well known (Gantmakher, 2010; Golub and Loan, 1996), the spectral matrix norm is equal to maximum singular value of the matrix, and the spectral norm of inversion of the matrix is equal to inversion of minimum singular value of the matrix

$$C\{\mathbf{M}\} = \|\mathbf{M}\| \cdot \|\mathbf{M}^{-1}\| = \alpha_M\{\mathbf{M}\} \cdot \alpha_m^{-1}\{\mathbf{M}\}, \quad (28)$$

where $\alpha_M\{\mathbf{M}\}, \alpha_m\{\mathbf{M}\}$ are maximum and minimum singular values of matrix $\mathbf{M}$ respectively. If dependency of the vector of parameters $\mathbf{q}$ for matrix $\mathbf{M}$ is taken into account, equation (28) takes the form $C\{\mathbf{M}\} = \alpha_M\{\mathbf{M}(\mathbf{q})\} \cdot \alpha_m\{\mathbf{M}(\mathbf{q})\}$. Then

$$\left.\frac{\partial C\{\mathbf{M}\}}{\partial q_\nu}\right|_{\mathbf{q}_0} = \left.\frac{\partial \alpha_M\{\mathbf{M}\}}{\partial q_\nu}\right|_{\mathbf{q}_0} \cdot \alpha_m^{-1}\{\mathbf{M}\} -$$

$$\alpha_M\{\mathbf{M}\}\alpha_m^{-2}\{\mathbf{M}\} \left.\frac{\partial \alpha_m\{\mathbf{M}\}}{\partial q_\nu}\right|_{\mathbf{q}_0}. \quad (29)$$

The parametric sensitivity function $\Theta_\nu(k,\mathbf{q})$ can be calculated, if the sensitivity functions of singular values of the matrix of eigenvectors and the sensitivity functions of maximum eigenvalue of the state matrix of system (25) are obtained. Full increment $\Delta\sup\{\|\mathbf{x}(k,\mathbf{q})\|\}$ of the upper bound (8) of the process by the norm of the state vector of system (25) is defined as

$$\Delta\sup\{\|\mathbf{x}(k,\mathbf{q})\|\} = \sum_{\nu=1}^{p}\Theta_\nu(k,\mathbf{q}_0)\Delta q_\nu = \Theta^T(k,\mathbf{q}_0)\Delta\mathbf{q}, \qquad (30)$$

where $\Theta^T(k,\mathbf{q}_0) = row\{\Theta_\nu(k,\mathbf{q}_0)\}; \nu = \overline{1,p}$. Obviously, the varying upper bound represents a composition of (8) and (30).

Based on the obtained equations, the following algorithm for the robustness estimation of large deviations in linear discrete-time system with parametric uncertainties is proposed. The algorithm assumes that parameters of the system are linearly dependent on the uncertainties and has the following steps.

1. Define a discrete-time system with uncertainties of parameters in form (25).
2. Calculate sensitivity matrix $\left.\frac{\partial\bar{\mathbf{F}}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}$ to variation of parameter q_v.
3. Calculate derivative of the maximum eigenvalues of the state matrix of system according to
$$\left.\frac{\partial\bar{\lambda}_{\max}}{\partial q_v}\right|_{\mathbf{q}_0} = \left(\mathbf{M}^{-1}\left.\frac{\partial\bar{\mathbf{F}}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}\mathbf{M}\right)_{ii} \qquad (31)$$
4. Calculate sensitivity functions of maximum and minimum singular values of the eigenvectors matrix in relation to the next forms respectively
$$\frac{\partial}{\partial q_v}\alpha_M\{\mathbf{M}(\mathbf{q})\}|_{\mathbf{q}_0} = \left(\mathbf{U}^T\left.\frac{\partial\mathbf{M}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}\mathbf{V}\right)_{11}, \qquad (32)$$
$$\frac{\partial}{\partial q_v}\alpha_m\{\mathbf{M}(\mathbf{q})\}|_{\mathbf{q}_0} = \left(\mathbf{U}^T\left.\frac{\partial\mathbf{M}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}\mathbf{V}\right)_{nn}, \qquad (33)$$
where $\mathbf{U}$ and $\mathbf{V}$ are left and right singular basis of singular value decomposition respectively
$$\mathbf{M} = \mathbf{U}\left\{\Sigma = diag\left(\alpha_i; i=\overline{1,n}\right)\right\}\mathbf{V}^T. \qquad (34)$$
Matrix $\frac{\partial\mathbf{M}(\mathbf{q})}{\partial q_v}$ consists of sensitivity functions of eigenvectors
$$\left.\frac{\partial\mathbf{M}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0} = \text{row}\left\{\left.\frac{\partial\mathbf{M}_i(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}; i=\overline{1,n}\right\}, \qquad (35)$$
where
$$\left.\frac{\partial\mathbf{M}_i(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0} = \sum_{k=1,k\neq i}^{n}\gamma_{ik}^{v}\mathbf{M}_k; \gamma_{ii}^{v} = 0 \qquad (36)$$
and coefficients γ_{ik}^{v} can be obtained as
$$\gamma_{ik}^{v} = \frac{\left(\mathbf{M}^{-1}\left.\frac{\partial\bar{\mathbf{F}}(\mathbf{q})}{\partial q_v}\right|_{\mathbf{q}_0}\mathbf{M}\right)_{ik}}{\bar{\lambda}_i - \bar{\lambda}_k}; k\neq i. \qquad (37)$$
5. Calculate the derivative of conditional number of the eigenvectors matrix in form (29).
6. Form parametric sensitivity function of the upper bound (27) of large deviation in discrete-time system (25).
7. Find full increment of the upper bound of the process according to (30).
8. Establish the curves of dependent of full increment of the upper bound of the process on discrete time.

Note that a full increment $\Delta\bar{\lambda}_{\max}(\mathbf{q})$ of maximum eigenvalue can be calculated according to its sensitivity function (31) such that

$$\Delta\bar{\lambda}_{\max} = \sum_{\nu=1}^{p}\left.\frac{\partial\bar{\lambda}_{\max}}{\partial q_\nu}\right|_{\mathbf{q}_0}\Delta q_\nu \qquad (38)$$

Then the full increment (30) taking into account sensitivity function (27) and the full increment of the maximum eigenvalue (38) can be defined as

$$\Delta\sup\{\|\mathbf{x}(k,\mathbf{q})\|\} = \sum_{\nu=1}^{p}\Theta_\nu(k,\mathbf{q}_0,\Delta\bar{\lambda}_{\max})\Delta q_\nu, \qquad (39)$$

where

$$\Theta_\nu(k,\mathbf{q}_0,\Delta\bar{\lambda}_{\max})\,|_{\|\mathbf{x}(0)\|=1} = \left.\frac{\partial C\{\mathbf{M}(\mathbf{q})\}}{\partial q_\nu}\right|_{\mathbf{q}_0}(\bar{\lambda}_{\max}+\Delta\bar{\lambda}_{\max})^k + C\{\mathbf{M}(\mathbf{q})\}k(\bar{\lambda}_{\max}+\Delta\bar{\lambda}_{\max})^{k-1}\left.\frac{\partial\bar{\lambda}_{\max}(\mathbf{q})}{\partial q_\nu}\right|_{\mathbf{q}_0}. \qquad (40)$$

Finally an upper bound $\sup\{\|\mathbf{x}(k,\mathbf{q})\|\}$ of the process $\|\mathbf{x}(k,\mathbf{q})\|$ is estimated by (8) and (40) as follows

$$\sup\{\|\mathbf{x}(k,\mathbf{q})\|\}|_{\|\mathbf{x}(0)\|=1} = C\{\mathbf{M}\}(\bar{\lambda}_{\max}+\Delta\bar{\lambda}_{\max})^k + \Delta\sup\{\|\mathbf{x}(k,\mathbf{q})\|\}. \qquad (41)$$

5 ROBUSTNESS ESTIMATION OF DEVIATION UNDER UNCERTAINTY OF DELAY IN CONTROL

Suppose that q is variation of delay

$$q = \tau = \tau_0 + \Delta\tau, \qquad (42)$$

where $\tau_0 = 0$; $\Delta\tau = (0; \Delta t]$. Then variations of large deviations can be estimated according to the proposed algorithm. Using the algorithm we can estimate upper bound variation (8) depending from changes of control signal delay. The main point is to get sensitivity matrix of the state matrix of the discrete-time closed-loop system from section 3.

Consider the above two cases. For unpredictable delay we get following sensitivity matrix of state matrix (22)

$$\left.\frac{\partial \bar{\mathbf{F}}(\tau)}{\partial \tau}\right|_{\tau_0} = \left.\begin{bmatrix} -\frac{\partial \bar{\mathbf{B}}(\tau)}{\partial \tau}\bar{\mathbf{K}} & \frac{\partial \bar{\mathbf{B}}_1(\tau)}{\partial \tau} \\ 0 & 0 \end{bmatrix}\right|_{\tau_0} = \begin{bmatrix} \bar{\mathbf{A}}\bar{\mathbf{B}}\bar{\mathbf{K}} & \bar{\mathbf{A}}\bar{\mathbf{B}} \\ 0 & 0 \end{bmatrix}. \quad (43)$$

For predictable delay the sensitivity matrix of state matrix (24) takes the form

$$\left.\frac{\partial \tilde{\mathbf{F}}(\tau)}{\partial \tau}\right|_{\tau_0} = \left.\begin{bmatrix} -\frac{\partial \bar{\mathbf{B}}(\tau)}{\partial \tau}\tilde{\mathbf{K}}_x & \frac{\partial \bar{\mathbf{B}}_1(\tau)}{\partial \tau} - \frac{\partial \bar{\mathbf{B}}(\tau)}{\partial \tau}\tilde{\mathbf{K}}_\chi \\ 0 & 0 \end{bmatrix}\right|_{\tau_0} = \begin{bmatrix} \bar{\mathbf{A}}\bar{\mathbf{B}}\tilde{\mathbf{K}}_x & \bar{\mathbf{A}}\bar{\mathbf{B}}(1+\tilde{\mathbf{K}}_\chi) \\ 0 & 0 \end{bmatrix}. \quad (44)$$

According to the algorithm, these matrices are the basis for next ensuing calculations of sensitivity functions of eigenvalues (31), eigenvectors (36), singular values (32), (33) and as result the upper bound (41).

6 EXAMPLE

Consider a stable continuous plant (9) with matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.01 & -0.2 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let us design a discrete-time system for the plant with sample time $\Delta t = 0.01s$ using modal control with desired eigenvalues $\sigma(\bar{\mathbf{F}}) = \{0.9802, 0.9139\}$. The discrete-time system without control signal delay is described. There is a deviation of free motion of the system, that is shown on the Fig.4, where deviation achieves $\max \|x(k)\|$=1.536. Note, in this example the norm of the vector of initial conditions equals one $\|x(0)\| = 1$ for all cases. Then let us consider the above two cases of presence of delay in control signal.

6.1 The Case of Unpredictable Delay of Control Signal

For the case of unpredictable control signal delay the system (21) is described with matrices (22). The Fig.5 shows norms of free motion for delays $\tau = 0.1\Delta t$ (curve 1),$\tau = 0.3\Delta t$ (curve 2) and the upper bound (41) for $\tau = 0.3\Delta t$.

The deviation achieves $\max \|x(k)\| = 20$. It should be noted the system becomes unstable for $\tau > 0.45\Delta t$.

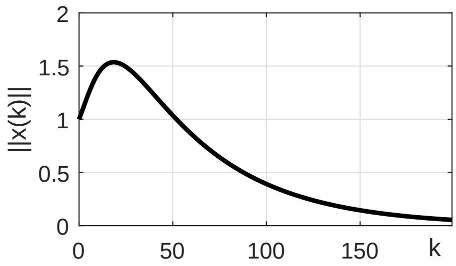

Figure 4: Norm of free motion of the discrete-time system without control signal delay.

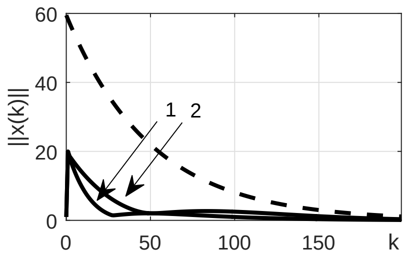

Figure 5: Norm of free motion and upper bound estimation of the discrete-time system with unpredictable delay.

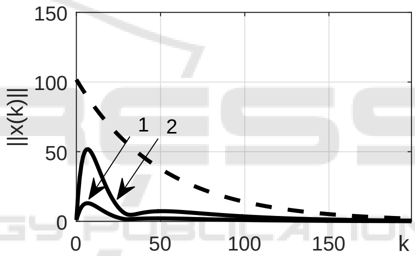

Figure 6: Norm of free motion and upper bound estimation of the discrete-time system with predictable delay.

6.2 The Case of Predictable Delay of Control Signal

For the case of predictable control signal delay the system (21) is described with matrices (24) and additional desired eigenvalue for control law (23) is $\bar{\lambda}_3 = 0.8187$. The Fig.6 shows norm of free motion for delays $\tau = 0.1\Delta t$ (curve 1),$\tau = 0.3\Delta t$ (curve 2) and the upper bound (41) for $\tau = 0.3\Delta t$.

The deviation achieves $\max \|x(k)\| = 51.8$ and the system stays stable. Obviously, the value of the delay in both cases affects the level of deviation in free motion of the discrete-time system. Moreover, the predictable control signal delay has biggest influence on the level of deviation than unpredictable control signal delay, but stability is guaranteed.

7 CONCLUSIONS

The aim of the paper was to get estimation of robustness of large deviations in free motion of stable linear discrete-time systems with parametric uncertainties. The tracking discrete-time system with predictable control signal delay and unpredictable control signal delay was considered in the paper. Using a combination of state-space approach and the sensitivity theory methods the estimation robustness of the large deviations was obtained. It was derived that the upper bound by the norm of the large deviations in linear discrete-time systems with parametric uncertainties depends of sensitivity functions of singular values of the eigenvectors matrix of the system state matrix. At the same time the sensitivity matrix of a state matrix depends on the value of the control signal delay, and that relationship was obtained. The algorithm for robustness estimation of the large deviations was proposed and the illustrative example was given. It was shown, that the predictable control signal delay has biggest influence on the level of deviation than unpredictable control signal delay, but stability is guaranteed.

In future, it is supposed to expand the results of the paper to the case of a discrete-time systems with parametric uncertainties having complex eigenvalues.

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