Entropy Map Might Be Chaotic

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Abstract: Chaos is a phenomenon observable in many areas. Chaotic behaviours can be visualized in chaotic maps, which are deterministic iterative functions and sensitive to initial conditions. As a result, they are wildly adopted in random number generator, image encryption, etc. In this paper, two new chaotic maps inspired by information entropy are proposed. Through bifurcation diagram and Lyapunov exponent analysis, period doubling bifurcations are observed and chaos is suggested. Furthermore, these maps lead to a special case of the Frobenius-Perron operator in their distributions and are extended to the complex plane to obtain the Julia set.

1 INTRODUCTION

Chaos is a nonlinear phenomenon in the physical world. First proposed by Lorenz (Lorenz, 1963), a chaotic system is a deterministic system sensitive to initial conditions: a small change at the beginning can magnify into large variations in the long term.

Chaotic maps are iterative functions in dynamics systems that exhibit chaotic behaviour for special parameters of the related function. They can be classified as discrete or continuous for real or complex variables. The Lorenz system, for instance, is a continuous chaotic map.

One-dimensional chaotic maps are discrete chaotic maps. They became popular research areas since the discovery of the Logistic Map in 1976. May discovered that these simple mathematical models could lead to complicated dynamics (May, 1976). Afterwards, more chaotic maps had been found, including classical maps like Tent Map (Devaney, 1984), Sine Map (Strogatz, 1994), and Doubling Map (Hirsch et al., 2013). Chaotic maps are useful in random number generator and image encryption due to their deterministic properties and high sensitivities to initial conditions.

In recent years, numerous additional chaotic maps have been proposed and analysed. Alpar constructed a simple fraction in a square map with one variable and two parameters, and studied this map through stability bifurcation, Lyapunov exponents, and cobweb plot analysis (Alpar, 2014). A novel one-dimensional sine powered chaotic map was proposed and applied in a new image encryption scheme by Mansouri et al. (2020). Lambić proposed a new discrete chaotic map according to the composition of permutations (Lambić, 2015).

When discrete chaotic maps involve complex variables, they can be represented as Julia sets. In general, a Julia set is a fractal in the complex plane (Peitgen et al., 2004) defined as the following (Falconer, 2014): First, take $f: \mathrm{C} \rightarrow \mathrm{C}$ as mapping function with complex parameter. Usually, f^{k} is the k composition $f \circ \cdots \circ f$, and $f^{\mathrm{k}}(\omega)$ is the k-th iteration $f(f(\cdots(f(\omega))\cdots))$. Then, the filled-in Julia set becomes:

$$K(f) = \{z \in C: f^{k}(z) \nrightarrow \infty\} \tag{1}$$

The Julia set of f is the boundary of filled-in Julia set, $J(f) = \partial K(f)$. If every neighbourhood of z exists different points of ω and υ, such that $f^{k}(\omega) \rightarrow \infty$, and $f^{k}(\upsilon) \nrightarrow \infty$, then z belongs to the Julia set $J(f)$.

Julia sets have a number of applications in the arts, computer science, and finance. For example, Cui et al. extended the Black-Scholes model to find the fractal in the model, which was a function for pricing European option (Cui et al., 2016).

In this research, we propose two new iterative functions based on the information entropy formula. The main goal of this paper is to show that they are

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chaotic through bifurcation diagram and Lyapunov exponent analysis. The rest of the paper is organized as follows. The new chaotic maps are proposed in Section 2. In Section 3, we verify these two maps are chaotic through bifurcation diagram and Lyapunov exponent. Their distributions are also analysed due to the emergence of an interesting phenomenon. In Section 4, we extend the new chaotic maps to complex plane to obtain the Julia sets. Finally, Section 5 concludes the paper with discussions.

2 NEW ONE-DIMENSION CHAOTIC MAPS

Information Entropy was introduced by Shannon in his famous paper "A Mathematic Theory of Communication", where he estimated the uncertainty of random variable (Shannon, 1948):

$$H = -K \sum_{i=1}^{n} p_i log(p_i) \quad (2)$$

where H denote information entropy, and K is a positive constant.

Based on the above information entropy formula, we propose two new iterative functions (eq.3 and eq.4):

$$x_{n+1} = -\alpha x_n ln x_n \quad (3)$$

where n is the iteration number, α the control parameter, $x_n \in [0, 1]$, and $\alpha \in (0, e]$. And

$$x_{n+1} = -\alpha(x_n ln x_n + \overline{x_n} ln \overline{x_n}) \quad (4)$$

where, $\overline{x_n}$ denotes $(1 - x_n)$, α the control parameter, $x_n \in [0, 1]$, and $\alpha \in (0, \frac{1}{ln2}]$.

3 ANALYSIS

3.1 Bifurcation Diagram

Bifurcation diagram is used to analyse the behaviour of chaotic map, which plots possible long-term values of the dynamic system as a function of one of its parameters. Normally, there are period doubling bifurcation and "period of 3". Observance of the "period of 3" in the bifurcation diagram implies chaos (Li et al., 1975).

The bifurcation diagrams of eq.3 and eq.4 are given in fig.1-2 and fig.3-4, respectively. There are clear period doubling bifurcation and chaotic region on the right size with a few numbers of periodic windows on the left. In the bifurcation diagrams fig.2 and fig.4, the window of "period of 3" can be clearly observed.

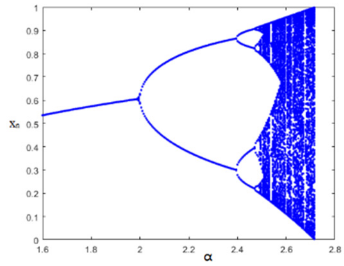

Figure 1: Bifurcation diagram of eq.3.

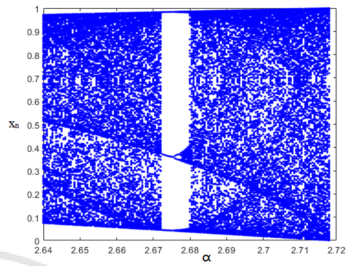

Figure 2: "Period of 3" of eq.3.

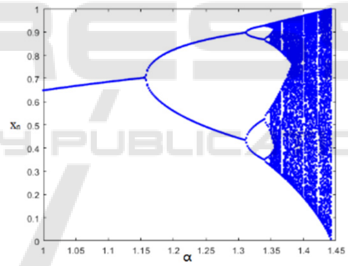

Figure 3: Bifurcation diagram of eq.4.

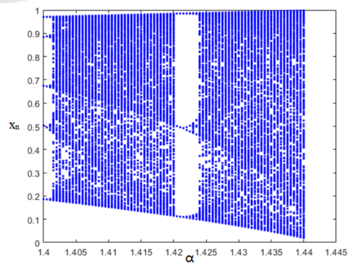

Figure 4: "Period of 3" of eq.4.

3.2 Lyapunov Exponent

The Lyapunov exponent λ is a strong instrument to measure a system's sensitivity to slight changes in the initial condition. λ quantifies the average increment of an infinitely small error at the initial point. $\lambda > 0$

indicates that the dynamic system is sensitive to the initial condition; $\lambda = 0$ means the system is stable; and $\lambda < 0$ reflects that the system tends to stabilize. If the λ for a one-dimensional chaotic map is positive, chaos is implied (Hao, 1993).

According to Peitgen et al. (2004), λ can be calculated as follows:

$$\lambda = \frac{1}{n}\sum_{k=1}^{n} ln\left|\frac{E_k}{E_{k-1}}\right| \tag{5}$$

where n is the iteration number and E_k the error in the k-th iteration.

Fig.5 and fig.6 show that the largest λ of these two maps are around 0.014, which is relatively small. Results suggest that the chaotic maps of information entropy are less chaotic compared to other chaotic maps.

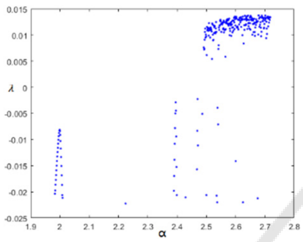

Figure 5: Lyapunov exponent of eq.3.

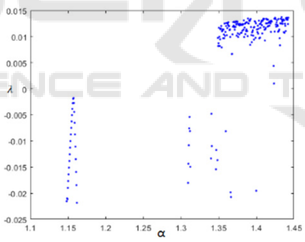

Figure 6: Lyapunov exponent of eq.4.

3.3 Distribution

Previous results indicate that eq.3 and eq.4 have chaotic regions. In the chaotic region, let parameter α equal to e in eq.6 and $\frac{1}{ln2}$ in eq.7.

$$x_{n+1} = -e x_n ln x_n \tag{6}$$

$$x_{n+1} = -\frac{1}{ln2}(x_n ln x_n + \overline{x_n} ln \overline{x_n}) \tag{7}$$

Fig.9 and fig.10 show the approximate distributions for eq.6 and eq.7. One interesting phenomenon is that the distribution of eq.7 is not symmetric while the function has an axis of symmetry around $x = 0.5$.

Here we show that the probability density function of eq.7 would not be symmetric if it is monotone in [0, 0.5]. Let y denote X_{n+1} and x denote X_n. Let $\upsilon(y)$ denote the probability density function of y and $\upsilon(x)$ denote the probability density function of x. Based on the Frobenius-Perron function (Peitgen et al., 2004):

$$\upsilon(y) = \frac{\upsilon(x)}{|f'(x)|} + \frac{\upsilon(1-x)}{|f'(1-x)|} \tag{8}$$

Assume that $\upsilon(x) = \upsilon(1-x)$, then:

$$\upsilon(y) = \frac{2\upsilon(x)ln2}{|ln\frac{1-x}{x}|} \tag{9}$$

Integrate from 0 to 1:

$$\int_0^1 \upsilon(y)dy = \int_0^1 \frac{2\upsilon(x)ln2}{|\ln\frac{1-x}{x}|}dx = 1 \tag{10}$$

$$\int_0^{0.5} \frac{\upsilon(x)ln2}{|\ln\frac{1-x}{x}|}dx = \frac{1}{4} \tag{11}$$

Using Chebyshev integral inequalities:

$$\int_0^{0.5} \upsilon(x)\,dx \int_0^{0.5} \frac{ln2}{|\ln\frac{1-x}{x}|}dx \leq \frac{1}{2}\int_0^{0.5} \frac{\upsilon(x)ln2}{|\ln\frac{1-x}{x}|}dx \tag{12}$$

We show that $\int_0^{0.5} \frac{\ln 2}{|\ln\frac{1-x}{x}|}dx$ should be no larger than $\frac{1}{4}$ while simple calculation shows it is. There is clearly a contradiction. So the probability density function would not be symmetric if it is monotone in [0, 0.5].

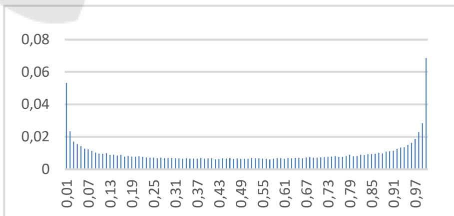

Figure 7: Distribution of eq.6.

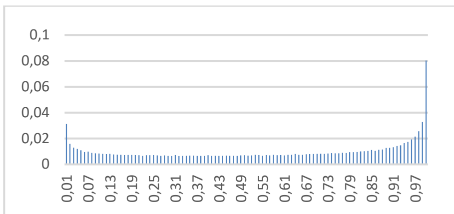

Figure 8: Distribution of eq.7.

4 JULIA SET

In this part, we calculate for the Julia set by constructing two iterative functions (eq.13 and eq.14) on the complex plane.

$$z_{n+1} = -z_n \ln z_n + \mathrm{c} \quad (13)$$

$$z_{n+1} = -z_n \ln(z_n + \mathrm{c}) \quad (14)$$

Fig.9 and fig.10 display the Julia sets for eq.12 and eq.13 with different values of c, respectively.

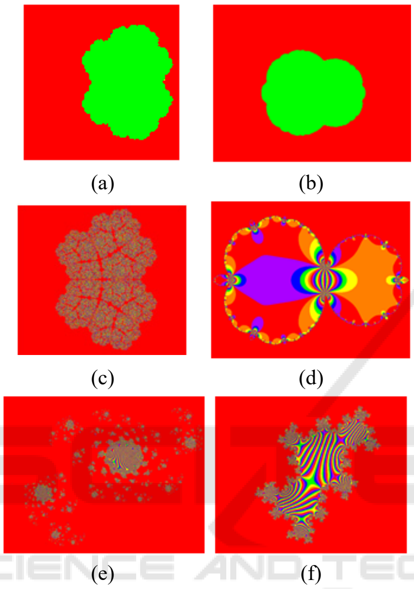

Figure 9: Julia set of eq.13. (a) c = 0; (b) c=0.75; (c) c = -0.15; (d) c=1; (e) c = 0.8+0.6i; (f) c=0.7i.

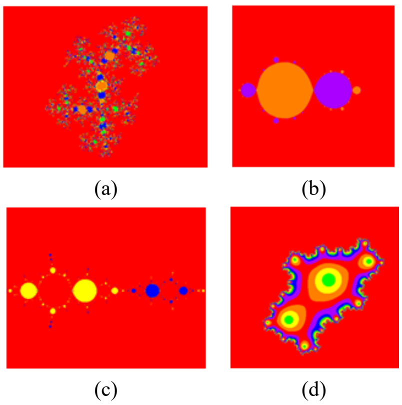

Figure 10: Julia set of eq.14. (a) c = -0.42i; (b) c=3; (c) c = 4; (d) c=2+1i.

5 CONCLUSIONS AND DISCUSSION

In this study, we propose two new chaotic maps, which are inspired by information entropy. Test and analysis results suggest that they are chaotic, with relatively small positive Lyapunov exponents around 0.014. In addition, we extend the chaotic maps to the complex plane and obtain the Julia sets.

In the distribution of eq.7, asymmetry seems to arise from a symmetry map. This might be caused by the computational software, or the map itself. This special Frobenius-Perron question remains unknown. Future work can attempt to calculate the exact distribution to answer this question and apply these chaotic maps and Julia sets to new applications in image encryption, finance, random number generation and other applications.

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