Similarity-inclusive Link Prediction with Quaternions

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Abstract: This paper proposes a Quaternion-based link prediction method, a novel representation learning method for recommendation purposes. The proposed algorithm depends on and computation with Quaternion algebra, benefiting from the expressiveness and rich representation learning capability of the Hamilton products. The proposed method depends on a link prediction approach and reveals the significant potential for performance improvement in top-N recommendation tasks. The experimental results indicate the superior performance of the approach using two quality measurements – hits rate, and coverage - on the Movielens and Hetrec datasets. Additionally, extensive experiments are conducted on three subsets of the Amazon dataset to understand the flexibility of this algorithm to incorporate different information sources and demonstrate the effectiveness of Quaternion algebra in graph-based recommendation algorithms. The proposed algorithms obtain comparatively higher performance, they are improved with similarity factors. The results show that the proposed quaternion-based algorithm can effectively deal with the deficiencies in graph-based recommender system, making it a preferable alternative among the other available methods.

1 INTRODUCTION

Recommender systems provide recommendations about various products and services to their users by applying other users' data. Their success is important for both users and e-commerce sites utilizing such systems. Providing accurate and dependable recommendations increases user satisfaction, in turn boosting the sales of products and services. Conversely, inaccurate, and unreliable product recommendations force users towards searching alternative sites for shopping. These systems are a challenging research field with many unresolved problems and many different hybrid recommendation algorithms proposed to overcome these problems. Graph-based hybrid models that use different information sources (text, images, ratings, etc.) for recommendation have been gaining more attention in recent years, (Yuan, 2012, Zhang, 2017, and Kurt, 2020).

Also, another key observation is that most studies in the recommendation algorithms have been mainly based on real-valued representations R, neglecting the rich potential of other spaces such as complex C and hypercomplex spaces H, (Zhang, 2019). This study investigates the concept of complex algebra and quaternion algebra, that are effectively established in the area of mathematics. Complex and hypercomplex representation learning methods are not only expanding the vector space also composing multiple spaces together. However, these spaces have tight links with associative retrieval, asymmetry, and learning latent inter-dependencies between components by multiplication of complex numbers or Hamilton products. The associative nature of complex representations going beyond multi-view representations is effectively developed in these studies (Danihelka, 2016, and Hayashi, 2017).

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Furthermore, the asymmetry of simple inner products in hypercomplex space (Trouillon, 2016, and Tay, 2018) yields a strong inductive bias to solving the asymmetrical problem of user-item matching. Since, the user and item embeddings are mainly belonged to a different class of entities.

Quaternion representations are based on hypercomplex numbers with three imaginary numbers. These representations are recently getting more attention and showing promise in real-world applications such as speech recognition (Trabelsi, 2017), image and signal processing (Witten, 2006, and Parcollet 2019). This is the same case with multi-view representations, however the latent components are connected by a complex-number system. Furthermore, the Hamilton product gives the interactions between imaginary and real numbers, enables an expressive blend of numbers that forms the final representation. Accordingly, the interaction function is also important to the recommendation system research area, it is evident that the Hamilton products are the proper option for user-item representation learning.

In (Zhang, 2019), novel recommendation algorithms in non-real spaces are proposed based on leveraging rich and expressive complex number multiplication or Hamilton products to compose user-item pairs. These proposed algorithms are called Complex collaborative filtering and Quaternion collaborative filtering (QCF), and they open up a new different way to apply collaborative filtering-based neural recommendation algorithm in non-real spaces. All in all, these approaches demonstrate the effectiveness of Quaternion algebra in recommender systems. The deep learning research area has seen significant improvement in the last decade; nevertheless, much of these works have been implemented in real-valued numbers. Recent studies show that a deep learning-based system utilizing complex numbers can be deeper for a fixed parameter budget regarding its real-valued counterpart. In (Gaudet, 2018), the benefits of generalizing one step further into the hyper-complex numbers, quaternions especially, are examined and yielded the framework of the deep quaternion networks. Moreover, the theoretical basis by reviewing quaternion convolutions, generating a new quaternion weight initialization design, and developing some algorithms for quaternion batch-normalization are introduced in (Gaudet, 2018).

Quaternion-based multi-valued architecture is introduced in some research fields, demonstrating that it has the potential with numerical examples of multi-channel prediction and classification (Greenblatt, 2018, and Saoud, 2017). A variety of real-valued learning frameworks have been presented in prior literature, hence multi-valued architecture is utilized in order to compensate for their drawbacks. However, a better way to represent multidimensional data is utilizing quaternions (Greenblatt, 2018). The motivation behind this representation is that a four-dimensional associative normed division algebra over the real numbers enables the multiplication and division of points in three-dimensional space (Greenblatt, 2018).

Furthermore, an adaptive method for a tag-rating based recommender system is introduced in (Yuan, 2012). A term-association matrix is represented to describe the relationship between the tags' and items' properties in this approach. Quaternions are used for the definition of the term-association matrix, and the components of this matrix are users, items, tags, and ratings, each a part of a quaternion number. A privileged matrix factorization method for CF by utilizing the quaternions is introduced in (Du, 2017). This method is utilized by review texts that are in companion with rating values to assist the learning of user and item factors/representation. This recommendation algorithm is also considered as a rating prediction problem based on the quaternions. Again, a user representation, an item representation, a rating, and a review are denoted as parts of a quaternion number in this algorithm (Du, 2017).

A novel graph-based recommendation algorithm depending on social networks is proposed by (Wang, 2010). This social network is developed between users and items, considering the information of ratings and tags. The users' co-tagging behaviours and the similarity relationship among these users are utilized by the graph to enhance the performance. This algorithm is also based on the Random Walk with Restarts method and yields a more natural and efficient way to represent social networks. Utilizing the similarity relationships and the tags make the adjacency matrix denser and improve the recommendation accuracy rate.

Rating conversion is implemented to generate an adjacency matrix based on the representation of complex numbers with real and imaginary parts in the Similarity Inclusive Link Prediction (SIMLP) and Complex Representation-based Link Prediction (CORLP) algorithms (Xie, 2015, Kurt, 2019 and 2020). In these algorithms, similar or dissimilar links were weighted by real numbers, whereas the like or dislike links were weighted by complex numbers (Xie, 2015, Kurt, 2019). The problem of recommendation generation is considered as a link prediction problem since the complex numbers yield a natural algebraic link among real and imaginary parts. Moreover, the available link prediction algorithms may be applied with the proposed SIMLP method and without any modifications.

In this paper, the proposed SIMLP algorithm is reformulated based on the representation of quaternion numbers with a scalar and imaginary vector part in the quaternion form. The similar valued links are denoted as a scalar part, and the dissimilar, like, and dislike valued links are denoted as the imaginary vector part of the quaternion. As a quaternion number provides a link between real and imaginary vector parts in the bipartite graph model, the problem of recommendation generation can still be considered as a link prediction problem. Besides that, the available link prediction algorithms can operate with the proposed quaternion-based recommendation method as in the SIMLP method.
With this goal in perspective, this paper presents a new quaternion-based graph framework for recommendation generation. Initially, we give a simple overview of the quaternions and a quaternion-based triangle closing model, and then utilize this model to generate a quaternion-based similarity-inclusive link prediction method in a graph structure. The remainder of the paper is organized as follows: The detailed representation of the proposed recommendation algorithms appears in Section 2. The evaluation measurements that are used in this study are given in Section 3. The application of the experiments in three real-world datasets and the discussion of the experimental results are included in Section 4. Finally, the results and future research directions are summarized.

1.1 Quaternions

The quaternions were first introduced by William Rowan Hamilton, and they are members of a noncommutative division algebra (Mishchenko, 2000). The formula of quaternion algebra can be mathematically stated as:

$$i^2 = j^2 = k^2 = ijk = -1, \tag{1}$$

The quaternions are just one example of a more general class of hyper-complex numbers proposed by Hamilton, and the set of quaternions is represented as H, H or Q_8.

Quaternions can be considered as an extension of complex numbers and operate in a four-dimensional space. It comprises of a real number and three imaginary numbers. By analogy with the complex form, complex numbers can be represented as a sum of real and imaginary parts, $a + i \cdot b$, hence a quaternion number can also be denoted as a linear combination of real and imaginary parts;

$$\mathrm{H} = a + b \cdot i + c \cdot j + d \cdot k \tag{2}$$

The Hamilton product of two Quaternions can be written as the products of the bases elements and the distributive law. Assume that two quaternions are given as $\mathrm{H}_1 = a_1 + b_1 \cdot i + c_1 \cdot j + d_1 \cdot k$ and $\mathrm{H}_2 = a_2 + b_2 \cdot i + c_2 \cdot j + d_2 \cdot k$, then the Hamilton product of them can be represented as follows:

$$\mathrm{H}_1 \otimes \mathrm{H}_2 = (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) \cdot i + (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) \cdot j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) \cdot k$$

It can be inferred that the multiplication of quaternions is both distributive and associative, but it is not commutative.

Moreover, H can be represented as: $\mathrm{H} = (w, v) = w + x \cdot i + y \cdot j + z \cdot k$, when w is real (scalar), and v is an imaginary (vector) part.

$$\mathrm{H} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ v \end{bmatrix} \begin{matrix} scalar\ \ part \\ vector\ \ part \end{matrix} \tag{3}$$

1.2 Quaternion-based Triangle Closing Model

In this paper, a quaternion-based triangle closing model is proposed depending on the graph models, which are introduced in (Harary, 1955 and 1967, Kunegis, 2012). Moreover, the new design of the model based on the social graph models is presented by (Kunegis, 2012). The extended version of this model is recommended based on the usage of other number systems to identify each edge/link, such as the quaternions or the complex number systems. The only possible relationship in a social graph depends on the friendship (Kunegis, 2012). Then, the social recommendation problem can be considered as recommending new friends depending on existing friendships. The fundamental model utilized for this purpose can be considered as the major law of the triangle closing model: people who have (possibly many) common friends can be all friends. Figure 1. (a) illustrates this principle of triangle closing model. Two adjacent friend links let us predict a new friend link; hence, "*The friend of my friend is my friend*" as a rule is given in Figure 1 (a). Another triangle closing principle in a social graph with friend and foe relationships is illustrated in Figure 1 (b). In such a social graph, new links can be inferred using the principle that can be stated as "*The enemy of my enemy is my friend*", (Kunegis, 2012). Moreover, these two principles can be converted for the user-item interaction graph by utilizing the user-user and item-item similar and dissimilar relationships. Hence, two adjacent similar links let us predict a new similar link. Furthermore, two adjacent dissimilar links let us predict a new similar link, illustrated in Figures 2 and 3.

The triangle closing model can be generated with four different combinations. First of all, the vertices of the triangle model may only be constructed with users' nodes, which means that the triangle model is generated with three user nodes. This triangle model has two types of relationships. For the user-user links, there is a similarity factor, $e_{similar}$ or $e_{dissimilar}$ between two individuals. This triangle model is illustrated in Figure 2. Similarly, the vertices of the triangle model may be generated with only item nodes, which means that the triangle model is generated with three-item nodes. In a similar manner, this triangle model has two different relationships. There is a similarity factor $e_{similar}$ or $e_{dissimilar}$ in this triangle model for the item-item links. This triangle model is illustrated in Figure 3.

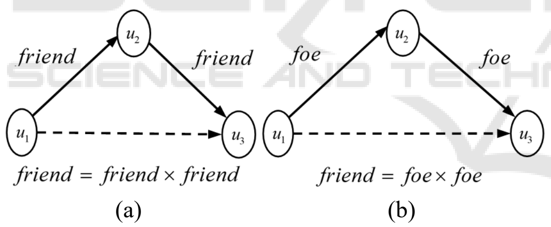


Figure 1: (a) Triangle closing model with only the friend relationship, (b) triangle closing model with friend and foe relationship.

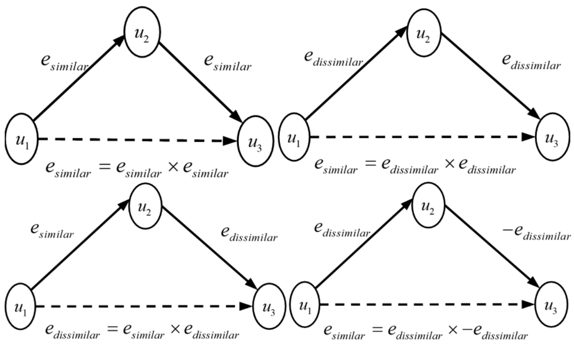


Figure 2: The triangle closing principle illustrated as the multiplication rule between similar/dissimilar relationships for only three user nodes.

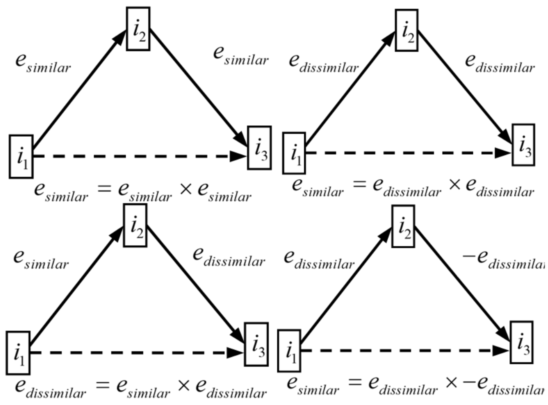


Figure 3: The triangle closing principle illustrated as the multiplication rule between similar/dissimilar relationships for only three item nodes.

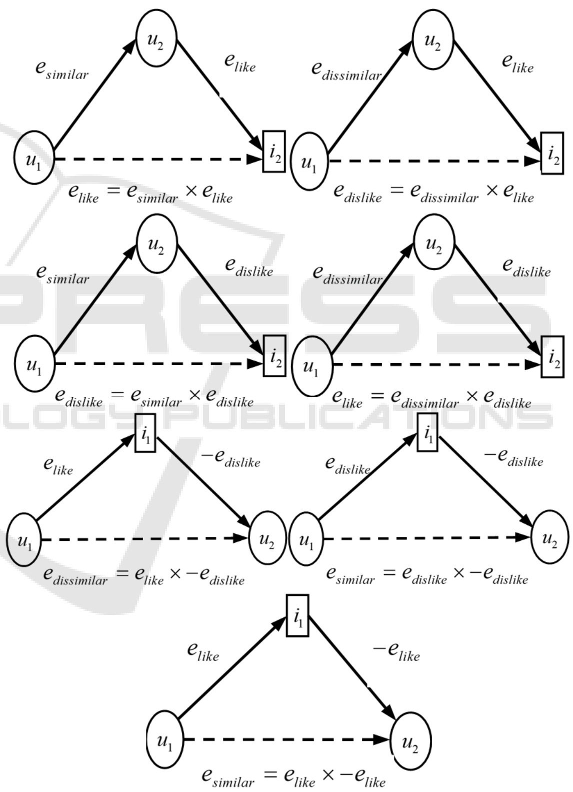


Figure 4: The triangle closing principle illustrated as the multiplication rule between similar/dissimilar and like/dislike relationships for two users' and an item' nodes.

Secondly, the vertices of the triangle model may be generated with two users nodes and an item node. For the user-item links, there is a similarity factor, e_{like} or $e_{dislike}$, between a user and item nodes. As a result of the necessity of recognizing the asymmetry between the item and the user, the triangle model includes item-user links. Then, there is a similarity factor, $-e_{like}$ or $-e_{dislike}$, between the item and user nodes. Subsequently, in the case of a link from user u to item i with the weight e_{like} or $e_{dislike}$, there is always a reverse link from item i to user u with a weight of $-e_{like}$ or $-e_{dislike}$. Moreover, there is a similarity factor, $e_{similar}$ or $e_{dissimilar}$, between two user nodes for the user-user links. This triangle model is illustrated in Figure 4.

Lastly, the vertices of the triangle model may be generated with a user node and two item nodes. Similarly, there is a similarity factor, e_{like} or $e_{dislike}$, for the user-item links and $-e_{like}$ or $-e_{dislike}$ for the item-user links between user nodes and item nodes. Furthermore, there is a similarity factor, $e_{similar}$ or $e_{dissimilar}$, between two item nodes for the item-item links. This triangle model is illustrated in Figure 5.

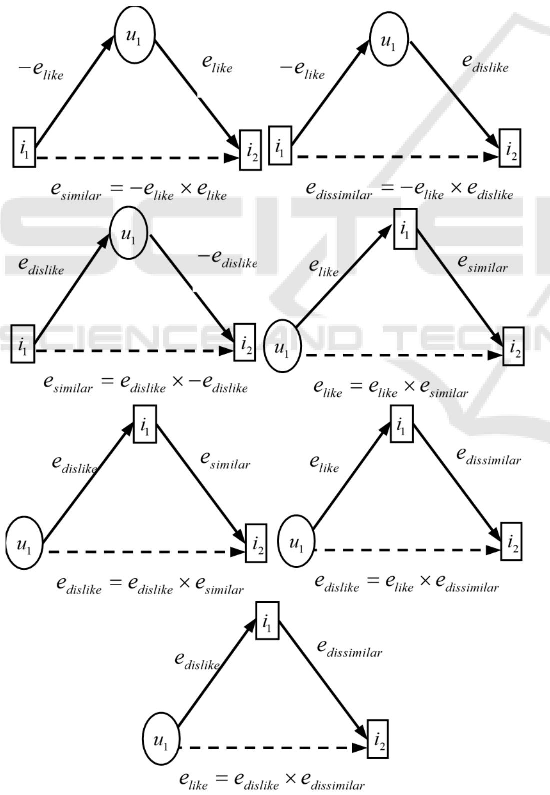


Figure 5: The triangle closing principle illustrated as the multiplication rule between similar/dissimilar and like/dislike relationships for two items and a user node.

In the quaternion-based triangle model, e_{like}, $e_{dislike}$ and $e_{similar}$, $e_{dissimilar}$ are normalized values just for the weights. This rule has four parts based on the triangle model comprising: three user nodes and their relations (see Figure 2), three-item nodes and their relations (see Figure 3), two users and an item node and their relations (see Figure 4), and finally a user node and two item nodes and their relations (see Figure 5). These are the major ideas of collaborative filtering from a different aspect, (Xie, 2015, Kurt, 2019). Since these multiplication principles of this model can be mathematically represented as follows:

$$\begin{aligned} e_{similar} &= -e^2_{like} = -e^2_{dislike} = -e^2_{dissimilar}, \\ e_{like} &= e_{similar} \cdot e_{like} = e_{dissimilar} \cdot e_{dislike}, \\ e_{dislike} &= e_{dissimilar} \cdot e_{like} = e_{similar} \cdot e_{dislike}, \\ e_{dissimilar} &= e_{like} \cdot -e_{dislike} = -e_{like} \cdot e_{dislike} \end{aligned} \tag{4}$$

Therefore, to solve this system of equations (Eq. 4), four different and nonzero constants need to be evaluated: $e_{similar}$, $e_{dissimilar}$, and e_{like}, $e_{dislike}$. Quaternion numbers provide an easy way to solve this system of equations when we set $e_{like} = i, e_{dislike} = j, e_{dissimilar} = k$ and $e_{similar} = 1$, where i, j, k are the imaginary unit vector. The requirements can be formalized as follows:

$$i^2 = j^2 = k^2 = ijk = -1 \text{ and } 1 = 1^2. \tag{5}$$

From this symbolization, a link has endpoints of the same type, and two items or two users may be weighted with a real number if there is a similarity factor $e_{similar}$. It means that the more similar the endpoints have the higher such value. A link has endpoints of the same type, among two users or two items, might be weighted with an imaginary weight k if there is a dissimilarity factor $e_{dissimilar}$. It means that the more dissimilar the endpoints are, the higher their value is. Besides that, a link with an imaginary weight can be a user-item or item-user link depending on the sign and interest. Such as, if a user u dislikes an item i, then the link is weighted with j from u to i, and the reversed link is weighted with $-j$ from i to u. Equivalently, if the user u likes the item i, then the link is weighted with i from u to i, and the reversed link is weighted with $-i$ from i to u. As opposed to similar links, we may categorize e_{like}, $e_{dislike}$ and $-e_{like}$, $-e_{dislike}$ only when the sign of the link's weight and the direction of the link are known at the same time. Since the sign of a similar link's weight is independent from the direction of the link, it can be concluded that the similar links provide the following rule:

$$e_{similar} = -e_{similar},$$
$$e_{dissimilar} = -e_{dissimilar}. \qquad (6)$$

1.3 Quaternion-based Adjacency Matrix

The adjacency matrix $\boldsymbol{A}$ is expanded as a quaternion matrix, and it can be mathematically formulated as:

$$\boldsymbol{A} = \boldsymbol{A}_{similar} + \boldsymbol{A}_{like} \cdot i + \boldsymbol{A}_{dislike} \cdot j + \boldsymbol{A}_{dissimilar} \cdot k, \qquad (7)$$

where the combination of item-item similarity $\boldsymbol{A}_{II}$ and user-user similarity matrices $\boldsymbol{A}_{UU}$ is denoted as $\boldsymbol{A}_{similar}$, the combination item-item dissimilarity $1_{n\times n} - \boldsymbol{A}_{II}$ and user-user dissimilarity matrices $1_{m\times m} - \boldsymbol{A}_{UU}$ are denoted as $\boldsymbol{A}_{dissimilar}$, and the user-item preference matrix is denoted as $\boldsymbol{A}_{UI}$ using both $\boldsymbol{A}_{like}$, $\boldsymbol{A}_{dislike}$ relationships. Moreover, the conjugate transpose of $\boldsymbol{A}_{UI}$ can be described in the same way as in (Kurt, 2019), $\boldsymbol{A}_{IU} = -\boldsymbol{A}^{T}{}_{UI}$. The preference matrices $\boldsymbol{A}_{like}$ $\boldsymbol{A}_{dislike}$ and the dissimilarity matrix $\boldsymbol{A}_{dissimilar}$ are complex matrices, while the similarity matrix $\boldsymbol{A}_{similar}$ is a real matrix.

The proposed Q-SIMLP algorithm differs slightly from the SIMLP-based recommendation method in the modeling of the adjacency matrix, and while calculating the powers of the adjacency matrix and providing the final recommendation in the same way. The user-user and item-item similarity and dissimilarity matrices of the user-item preference matrix are computed by utilizing cosine similarity measurement. After that, these similarity and dissimilarity factors are passed through from a threshold at 0.5. Then, the dissimilar links are multiplied by k, and these links are indicated as k in the imaginary part of the quaternions as stated in Eq. (7). Moreover, a user-item-like relational matrix is generated based on whether the rating is greater than 3, as stated in Eq. (7), while the user-item-dislike relational matrix is generated based on whether the rating is less than 3, as stated in Eq. (7). Following the summation of these matrices, the main adjacency matrix can be represented as in Eq. (7).

The components of quaternion-based adjacency matrix are mathematically stated as:

$$\boldsymbol{A}_{like} = \begin{pmatrix} 0 & \dots & 0 & r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{m1} & \cdots & r_{mn} \\ -r_{11} & \dots & -r_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -r_{m1} & \cdots & -r_{mn} & 0 & \cdots & 0 \end{pmatrix}, \boldsymbol{A}_{similar} = \begin{pmatrix} u_{11} & \dots & u_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mm} & 0 & \cdots & 0 \\ 0 & \dots & 0 & t_{11} & \dots & t_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n1} & \cdots & t_{nn} \end{pmatrix}$$

$$\boldsymbol{A}_{dislike} = \begin{pmatrix} 0 & \dots & 0 & r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{m1} & \cdots & r_{mn} \\ -r_{11} & \dots & -r_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -r_{m1} & \cdots & -r_{mn} & 0 & \cdots & 0 \end{pmatrix},$$

$$\boldsymbol{A}_{dissimilar} = \begin{pmatrix} 1-u_{11} & \dots & 1-u_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1-u_{m1} & \cdots & 1-u_{mn} & 0 & \cdots & 0 \\ 0 & \dots & 0 & 1-t_{11} & \dots & 1-t_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1-t_{n1} & \cdots & 1-t_{nn} \end{pmatrix} \qquad (8)$$

where u_{ij} denotes the similarity relationship between the i^{th} and j^{th} users, t_{ij} denotes the similarity relationship among the i^{th} and j^{th} items, r_{ij} expresses the like or dislike relationship among the i^{th} user and j^{th} item, and $-r_{ij}$ expresses the conjugate transpose of the like or dislike relationship between the i^{th} user and j^{th} item in Eq. (8). When r_{ij} expresses the like relationship between the i^{th} user and j^{th} item, r_{ij} is multiplied by i, which is an imaginary part of quaternions. Equivalently, if r_{ij} represents the dislike relationship between the i^{th} user and j^{th} item, r_{ij} is multiplied by j, again an imaginary part of quaternions. Moreover, $1-u_{ij}$ expresses the dissimilarity relationship between the i^{th} user and j^{th} users, and $1-t_{ij}$ expresses the dissimilarity relationship between the i^{th} item and j^{th} item as in Eq. (8). Equivalently, $1-u_{ij}$ and $1-t_{ij}$ are multiplied by k as an imaginary part of quaternions.

After the summation of these matrices, the main adjacency matrix $\boldsymbol{A}$ is built as in Eq. (9).

$$\boldsymbol{A}=\begin{pmatrix} u_{11} & \cdots & u_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mm} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n1} & \cdots & t_{nn} \end{pmatrix}+\begin{pmatrix} 0 & \cdots & 0 & r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{m1} & \cdots & r_{mn} \\ -r_{11} & \cdots & -r_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -r_{m1} & \cdots & -r_{mn} & 0 & \cdots & 0 \end{pmatrix}\cdot i+$$
$$\begin{pmatrix} 0 & \cdots & 0 & r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \ldots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{m1} & \cdots & r_{mn} \\ -r_{11} & \cdots & -r_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -r_{m1} & \cdots & -r_{mn} & 0 & \cdots & 0 \end{pmatrix}\cdot j+\begin{pmatrix} 1-u_{11} & \cdots & 1-u_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1-u_{m1} & \cdots & 1-u_{mn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1-t_{11} & \cdots & 1-t_{1n} \\ \vdots & \ddots & \vdots \ldots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1-t_{n1} & \cdots & 1-t_{nn} \end{pmatrix}\cdot k \quad (9)$$

Furthermore, this adjacency matrix is square, and eigenvalue decomposition can be applied to this matrix in Eq. (9). In the proposed quaternion-based method with another approach, the link prediction function can be multiplied by a parameter α; then, the predictions applied to $\boldsymbol{A}$ can be represented as:

$$P(\alpha\boldsymbol{A})=\lambda\cdot\alpha\boldsymbol{A}+\lambda_1\cdot(\alpha\boldsymbol{A})^3+\lambda_2\cdot(\alpha\boldsymbol{A})^5+\lambda_3\cdot(\alpha\boldsymbol{A})^7+\ldots \quad (10)$$

2 QUATERNION-BASED SIMILARITY-INCLUSIVE LINK PREDICTION METHOD

Rating conversion is necessary to generate the quaternion-based adjacency matrix in the proposed Q-SIMLP method, where the ratings/values in the user-item rating matrix are changed by imaginary numbers i or j based on whether the rating is greater than or equal to 3. In this sense, if the rating is less than 3, it is replaced with j, which means that the user expresses 'dislike' for the item; equivalently, imaginary value i is given to defining 'like', while the rating is greater than or equal to 3. Moreover, when the user-item pair (u,i) is not appeared in the training set, the corresponding component of the adjacency matrix is equal to zero. Following that, the user-user similarity and item-item similarity matrices of the preference matrix are generated by utilizing a cosine similarity measure to calculate the similarity values. On the other hand, we find the user-user dissimilarity and item-item dissimilarity matrices by utilizing the user-user similarity and item-item similarity matrices of the preference matrix, respectively, as stated in Eq. 6.8. Then, the components of the similarity and the dissimilarity matrices of the preference matrix are passed through a threshold at 0.5. These matrices include only binary values, with the similarity matrices represented as a scalar part of the quaternion-based adjacency matrix, as formulated in Eq. 8. The user-user dissimilarity and item-item dissimilarity matrices are multiplied by k, and these matrices are taken as one of the imaginary parts of the adjacency matrix. After generating the summation of the like relationships and dislike relationships matrices and the dissimilarity matrices, the entire imaginary part of the quaternion-based adjacency matrix is developed.

The evaluation of the powers of the quaternion-based adjacency matrix and providing the final recommendation follow the same procedure as the SIMLP algorithm for the proposed Q-SIMLP algorithm. The hyperbolic sine function is considered as a link prediction function for the proposed Q-SIMLP algorithm. Hence, the closest values among the nodes are evaluated by the power sum of the adjacency matrix, and the summation of each entry of the top-right and top-left components represents the degree of whichever item is relevant to a specific user. Following the summation of the odd powers of the adjacency matrix, the prediction scores that denote item recommendation to a particular user are obtained. These scores are denoted as the summation of a scalar/real part and the imaginary part i of the entire score. Since only the like relationships are taken into consideration for recommendation generation, the prediction scores are sorted in a descending order since the user will like the item if the score is positive, or will dislike the item when the score is negative. When the scores are positive and higher in value, such items will be recommended to a selected user as new and never-seen-before alternatives. Furthermore, top-N recommendation lists are produced for every user by these ranked prediction scores (Bedi, 2017).

2.1 Quaternion-based Hybrid Recommender System

The proposed quaternion-based hybrid recommendation algorithm differs slightly from the Q-SIMLP method in the modeling of the adjacency matrix. For the present system, the user-item ratings and visual images of the entire items in the datasets are known. Hence, the method benefits from such visualization by means of the AlexNet features, as mentioned in (amazon website). On the other hand, each users' visual feature vector is generated in accordance to their preferences. In the beginning, all items noticed, rated, or purchased by a user identified. Then, the AlexNet feature vectors of these items are extracted and summed up. Lastly, the summation mean is calculated using the number of items that users' either noticed, rated, or purchased before. Since each user can be represented as a 4096-dimensional visual feature vector, a user visual-feature matrix can be generated for each dataset. Following the generation of the user visual-feature and item visual-feature matrices, we can find the user-user and item-item similarity matrices by utilizing these feature matrices.

The quaternion-based adjacency matrix generation for the hybrid recommendation algorithm is modified in the same manner as the proposed Q-SIMLP algorithm. Also, the rating conversion part of the adjacency matrix follows the same procedures to generate the adjacency matrix. Following that, the user-user similarity matrix is generated from the user visual-feature matrix by applying cosine similarity measures to evaluate similarity values. Besides that, the item-item similarity matrix is generated from the item visual-feature matrix by applying the cosine similarity measures to compute the similarity values. In other aspects, the user-user and item-item dissimilarity matrices are generated by applying the user-user similarity and item-item similarity matrices, respectively. Then, the components of the similarity and dissimilarity matrices of the system are passed through a threshold at 0.5 since these matrices only consist of binary values. Similar to Q-SIMLP algorithm, we take the similarity matrices as a scalar part of the adjacency matrix, as formulated in Eq. (11). Also, the user-user and item-item dissimilarity matrices are multiplied by k and taken as one of the imaginary parts of the adjacency matrix. Following the summation of these matrices, the main adjacency matrix can be formed as in Eq. (11).

$$A = A_{visual\text{-}similar} + A_{like} \times i + A_{dislike} \times j + A_{visual\text{-}dissimilar} \times k \qquad (11)$$

This quaternion-based adjacency matrix is a square matrix. Therefore, the same link prediction (hyperbolic) function can be used on this adjacency matrix in Eq. (11) to evaluate the power sum of this matrix. In this way, the recommendation methodology adopted here is the same as in the Q-SIMLP recommendation algorithm.

An example of a user-item signed graph generation process for a quaternion-based hybrid recommender system is illustrated in Figure 6 and Figure 7. The user-item rating matrix and bipartite signed graph model of this rating matrix are drawn in Figure 6 (a). The green links represent '*like*' edges denoted as i, and the red links represent '*dislike*' edges denoted as j in the bipartite signed graph as in Figure 6 (a). The user-feature matrix and user-user relationship graph are drawn in Figure 6 (b). The green links represent user-user '*similar*' relationships, while the red links represent user-user '*dissimilar*' relationships in Figure 6 (b). The item-feature matrix and item-item relationship graphs are drawn in Figure 6 (c). Finally, the generated user-item signed graph for the quaternion-based hybrid recommender system is drawn in Figure 7.

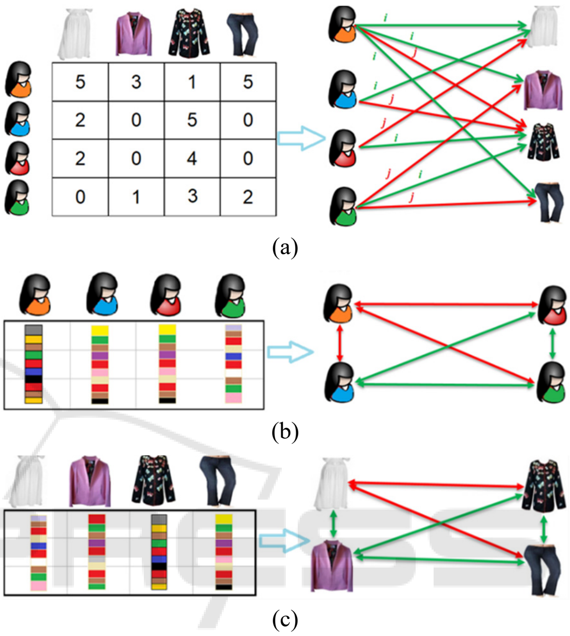


Figure 6: (a) User-item rating matrix and bipartite signed graph, (b) user-feature matrix and user-user relationship graph, (c) item-feature matrix and item - item relationship graph.

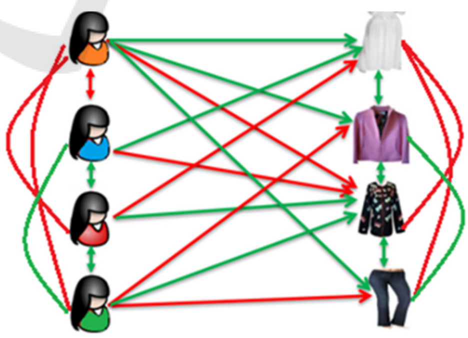


Figure 7: The generated user-item signed graph.

3 EXPERIMENTAL EVALUATION

The proposed Q-SIMLP algorithm, along with other methods, is applied on two real-world datasets for comparison: MovieLens (grouplens, website) and MovieLens Hetrec, (hetrec2011, website). First of all, rating conversion is applied to the user-item rating matrix in these datasets, they are converted into two imaginary parts, i and j, of the quaternions. Then, the cosine similarity measure is applied to the user-item rating matrices of these datasets to find the similarity values. Finally, the user-user and item-item similarity matrices of user-item rating matrices for these datasets are obtained after the cosine similarity values are passed through a threshold at 0.5 and 0.7 for Movielens and Hetrec datasets, respectively. Likewise, the user-user and item-item dissimilarity matrices of the user-item rating matrices for these datasets are obtained after the dissimilarity values are passed through a threshold at 0.5 and 0.3 for Movielens and Hetrec datasets, respectively. Also, the threshold of dissimilarity values is represented for these datasets. The threshold of similarity values for the Hetrec dataset is indicated as 0.7, since this dataset is sparser than the Movielens dataset. Following the combination of all these matrices, the major quaternion-based adjacency matrices are built as a square matrix for these two datasets as in Eq. (8). Hence, we can apply the hyperbolic sine function on the adjacency matrix as a link prediction function, as in (Xie, 2015, Kurt, 2019). Moreover, we multiply the link prediction function with a parameter α, since the predictions applied to $\boldsymbol{A}$ can be represented as:

$$\alpha \cdot sinh(\boldsymbol{A}) = sinh(\alpha \cdot \boldsymbol{A}) = \boldsymbol{U} \cdot (\alpha \Lambda) \cdot \boldsymbol{U}^T \qquad (12)$$

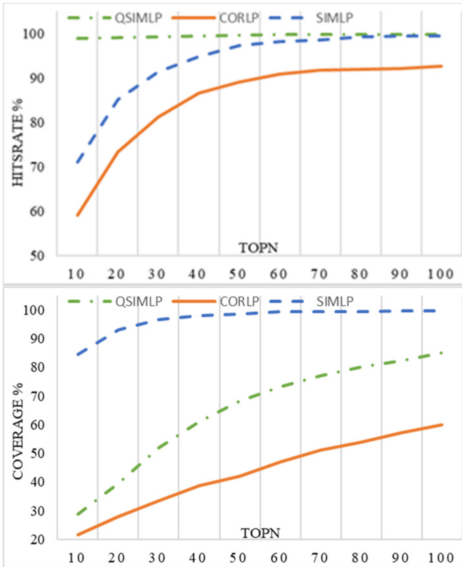
When the adjacency matrix is a square n by n matrix, the sum of the n eigenvalues of $\boldsymbol{A}$ is the same as/equivalent to the trace of $\boldsymbol{A}$;

$$\sum_{i=1}^{n} \lambda_i = trace(\boldsymbol{A}) \qquad (13)$$

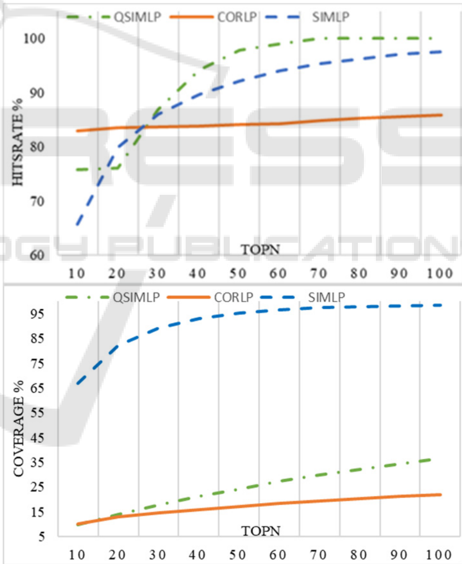
The theory and proof of Eq. (13) are given in the Appendix as theorem 2 in (Kurt, 2019, *phd thesis*). Furthermore, we assumed that the trace of the adjacency matrix is equal to the length of the adjacency matrix since all the components of adjacency matrix values (similar, dissimilar, like, and dislike values) are evaluated as binary values before the rating conversion as quaternion numbers. Then, the scaling parameter α is chosen as

$$\alpha = 1 / length(\boldsymbol{A}), \qquad (14)$$

since the largest eigenvalue cannot be bigger than the trace of the adjacency matrix. Hence, to normalize the eigenvalues of $\boldsymbol{A}$, we set α as in Eq. (14). Moreover, to evaluate the results of CORLP and SIMLP approaches, α is set as same as in the Q-SIMLP method.



(a)



(b)

Figure 8: Comparison of the Q-SIMLP, CORLP, and SIMLP methods by coverage and hits rate for the top-N recommendation on MovieLens (a) and Hetrec (b) datasets.

The testing methodology adopted in the proposed rating-based recommendation algorithm is the same as in these two former studies (Xie, 2015, Kurt, 2019). The ratings are divided by two subsets, for training and testing, for each dataset as in (Kurt, 2019). Also, the rating conversion threshold value is set as 2.5 for the Hetrec dataset, hence this dataset includes decimal rating numbers. The performance of the proposed Q-SIMLP recommendation method is measured by using the metrics, hits rate, and coverage. Figure 8 illustrates the comparison of the proposed Q-SIMLP, SIMLP, and CORLP recommendation algorithm with path length 3 for the top-N recommendation on the Movielens (a) and Hetrec (b) dataset. Figure 8 shows that the hits rate of the Q-SIMLP method is higher than the SIMLP and CORLP method. However, the coverage of the Q-SIMLP method is relatively less than the SIMLP method, and still more than the CORLP method on these two datasets. It can be seen in Figure 8 that the Q-SIMLP method can give better results for the top-10 recommendation task when compared to the results of the SIMLP and CORLP for the top-100 recommendation task. Also, the hits rate of the Q-SIMLP method for the top-100 recommendation task is higher than the hits rates of the SIMLP and CORLP methods for the same purpose. Hence the quaternion-based recommendation algorithm can reach more accurate results in a faster and easier way compared to the other approaches. It is concluded that the Q-SIMLP method provides accurate recommendations by consuming less time.

Another question to be addressed is whether the proposed Q-SIMLP approach that utilizes cosine similarities performs better than CORLP and SIMLP approaches for the top-N recommendation tasks. The hits rate and coverage are utilized as the evaluation metrics to measure the performance of the proposed Q-SIMLP recommendation algorithm. One-way Anova test is applied to further evaluate performance differences between Q-SIMLP and SIMLP and CORLP approaches, respectively. Thus, these special hypotheses examined in this paper are:

- H1: The Q-SIMLP-based recommendation approach obtains a higher hits rate than the SIMLP and CORLP approaches do.
- H2: The Q-SIMLP-based recommendation approach obtains higher coverage than the SIMLP and CORLP approaches do.

Table 1: The p-values of the comparison of the Q-SIMLP between CORLP and SIMLP methods regarding hits rate on MovieLens and Hetrec datasets.

Methods / Dataset	CORLP	SIMLP
Movielens	0.0005	0.0479
Hetrec	0.0137	0.0538

Table 1 has only one p-value that reflects no significant differences between the Q-SIMLP and SIMLP methods concerning hits rate on the Hetrec dataset. Since this p-value is very close to 0.05 (0.0538≅0.05), it can be concluded that there are statistically significant differences between CORLP, SIMLP, and Q-SIMLP methods concerning hits rate for the experiments on Movielens and Hetrec datasets. Table 2 indicates that there are statistically significant differences among CORLP, SIMLP, and Q-SIMLP methods concerning coverage for the experiments on Movielens and Hetrec datasets, hence all the p-values are smaller than 0.05. The hypotheses H1 and H2 are supported for each evaluation metric utilized in this paper.

Table 2: The p-values of the comparison of the Q-SIMLP between CORLP and SIMLP methods regarding coverage on MovieLens and Hetrec datasets.

Methods / Dataset	CORLP	SIMLP
Movielens	0.0087	0.00006
Hetrec	0.0249	5×10^{-11}

The proposed quaternion-based hybrid recommendation algorithm is implemented on three real-world Amazon datasets (amazon website): Cell phone, Beauty, and Clothing. These datasets are introduced in (Zhang, 2017). As the same process in the Q-SIMLP algorithm, quaternion-based rating conversion is applied to the user-item rating matrices in these datasets. Then, the cosine similarity measure is applied to user visual-feature and item visual-feature matrices of these datasets. Thus, the user-user and item-item similarity matrices of user-item rating matrices for these datasets are obtained after the cosine similarity values are passed through a threshold at 0.6, 0.7, and 0.6 for Cell phone, Beauty, and Clothing datasets, respectively. Similarly, the user-user and item-item dissimilarity matrices of user-item rating matrices for these datasets are reached after the dissimilarity values are passed through a threshold at 0.4, 0.3, and 0.4 for Cell phone, Beauty, and Clothing datasets, respectively. At the same time, the threshold of dissimilarity values depends on these values as observed in these datasets. Accordingly, the one for the Beauty dataset is the highest since this dataset is sparser than the others.

Following the combination of all these matrices, the main quaternion-based adjacency matrices are generated as a square matrix for these three datasets as in Eq. (11) since the hyperbolic sine function can be applied to the adjacency matrix as a link prediction function, as in (Kurt, 2019). Next, we multiply the hyperbolic sine function by a scaling parameter α, as introduced in the Q-SIMLP algorithm.

The testing methodology adopted in the quaternion-based hybrid recommendation algorithm slightly alternates from the other hybrid-SIMLP recommendation method that is introduced in (Kurt, 2020). Three product categories of different sizes and density levels are adopted, along with the standard 10-core datasets generated from each 5-core dataset, for the experiments. The density level of a dataset is calculated as in (Zhang, 2017);

$$sparsity = \frac{\#zero\ elements}{\#total\ elements},\ density = 1 - sparsity \tag{15}$$

in which $\#zero\ elements$ is denoted as the number of zero values in the user-item rating matrix of a dataset, and the total number of elements in this matrix is denoted as $\#total\ elements$.

Table 3: Statistics of the 10-core datasets.

Datasets	#Users	#Items	#Interactions	Density
Clothing	5197	4248	37515	0.3%
Cell Phones	3214	2743	34083	0.39%
Beauty	5123	4774	74497	0.17%

Table 4: The performance comparison of Q-Hybrid and Hybrid-SIMLP methods for the top-10 recommendation.

Datasets	Measures / Methods	Recall (%)	Hit Ratio (%)	Precision (%)
Beauty	Hybrid-SIMLP	29,21	61,73	2,92
	Q-Hybrid	**32,15**	**75,80**	**3,22**
Clothing	Hybrid-SIMLP	22,25	42,30	2,23
	Q-Hybrid	**36,84**	**65,18**	**3,68**
Beauty	Hybrid-SIMLP	29,75	57,59	2,98
	Q-Hybrid	**30,78**	**66,58**	**3,08**

Firstly, the user-item rating matrix of the 5-core data is filtered out for each user that has at least 10 ratings to generate a temporary 10-core dataset. Secondly, the temporary 10-core dataset is further filtered out for each item that has at least 5 ratings. The remaining items in the temporary 10-core data, which do not have 5 ratings, are omitted from the temporary dataset set for the generation of the final 10-core dataset. Since, the 10-core data is a subset of the 5-core data (Zhang, 2017), in which all users have at least 10 ratings and the items have at least 5 ratings. The statistics of the 10-core datasets are shown in Table 3.

The ratings are divided by two subsets as in the former experimental methodology. The test set includes only 5-star ratings and only items that are relevant to the corresponding users. The detailed procedure applied to produce the test and the training sets is the same as mentioned in (Zhang, 2017). Also, the performance of the quaternion-based hybrid recommendation algorithm is measured by using the metrics, hit-ratio, precision, and recall in the same way as in (Zhang, 2017). The results of the proposed quaternion-based hybrid recommendation (Q-Hybrid) method utilizing with top-10 recommendation tasks are demonstrated in Table 4. The results demonstrate that the Q-Hybrid recommendation algorithm obtains a higher hit-ratio, precision, and recall than other Hybrid-SIMLP recommendation algorithms on the Beauty, Cell Phone and, Clothing datasets. It is concluded that quaternion-based representations yield improvements for the performance of hybrid recommendation algorithms.

Moreover, the comparison results of the proposed Q-Hybrid approach with the Hybrid-SIMLP are discussed in terms of significance. In detail, whether the proposed Q-Hybrid approach performs better than the Hybrid-SIMLP approach for the top-N recommendation task. The range of N is taken from 10 to 100 for experiments on Cell Phone, Clothing, and Beauty datasets. The hit-ratio, recall, and precision are used as the evaluation metrics to measure the performance of the proposed Q-Hybrid and Hybrid-SIMLP recommendation algorithms. After that, the two-factor Anova test is employed to evaluate the performance differences between the two methods (Huang, 2002). Thus, the specific hypotheses analyzed in this paper are:

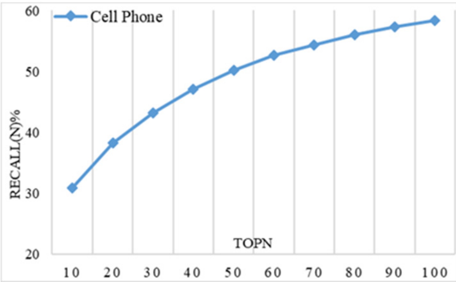
- H1: The Q-Hybrid recommendation approach obtains a higher hit-ratio than the Hybrid-SIMLP recommendation approach does.
- H2: The Q-Hybrid recommendation approach obtains higher recall than the Hybrid-SIMLP recommendation approach does.
- H3: The Q-Hybrid recommendation approach obtains higher precision than the Hybrid-SIMLP recommendation approach does.

Table 5: The p-values of the comparison among Q-SIMLP and Hybrid-SIMLP methods regarding to hit-ratio, recall, and precision on Cellphone, Beauty, and Clothing datasets.

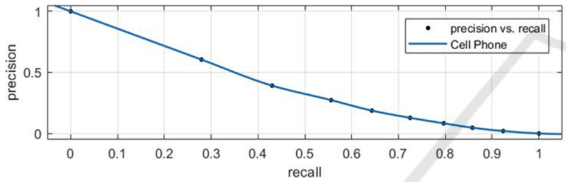
Measures / Datasets	Hit-Ratio	Recall	Precision
Cellphone	0.0002	0.5543	0.8507
Beauty	0.00001	0.0160	0.9126
Clothing	0.0008	0.0011	0.0588

Table 5 indicates that there are statistically significant differences between Q-Hybrid and Hybrid-SIMLP methods concerning hit-ratio for the experiments on Cell Phone, Beauty, and Clothing datasets. H1 is supported by the experimental results on each dataset defined in this paper. It can be seen in Table 5 that there are statistically significant differences between Q-Hybrid and Hybrid-SIMLP methods concerning recall for the experiments on

Beauty and Clothing datasets, not for the tests on the Cell Phone dataset. Moreover, H2 is supported only for the experiments on Beauty and Clothing datasets. Besides that, it can be concluded that there are no statistically significant differences between Q-Hybrid and Hybrid-SIMLP methods concerning precision for the experiments on each dataset. Finally, H3 is not supported by the experimental results on each dataset.

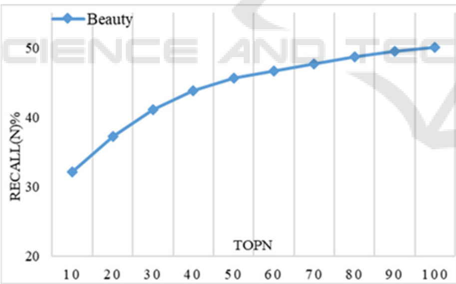


(a)

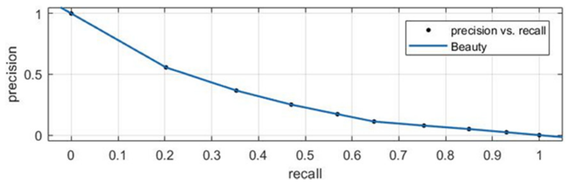


(b)

Figure 9: Cell Phone: (a) recall(N) and (b) precision-versus-recall on all items.



(a)

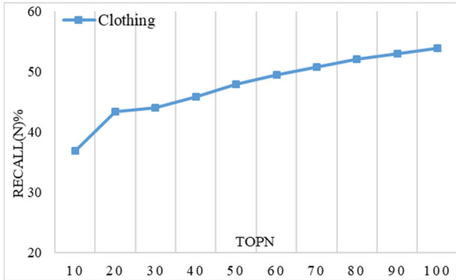


(b)

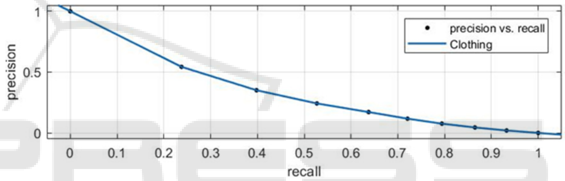
Figure 10: Beauty: (a) recall(N) and (b) precision-versus-recall on all items.

The recall(N) and precision(N) results of the proposed Q-Hybrid recommendation algorithm on Cell Phone, Beauty, and Clothing datasets are obtained and drawn respectively in Figure 9 (a), Figure 10 (a) and Figure 11 (a). Furthermore, the precision-versus-recall comparison of the results for each dataset are drawn in Figure 9 (b), Figure 10 (b), and Figure 11 (b). It can be seen from these figures that the precision and recall results of the Q-Hybrid method improve, compared to those of the Hybrid-SIMLP method, with increasing N for the top-N recommendation task.

Quaternion toolbox in Matlab (toolbox website), also known as 'qtfm_2.6', is used for the experiments to generate the quaternion-based adjacency matrix and to evaluate the hyperbolic sine of this matrix.



(a)



(b)

Figure 11: Clothing: (a) recall(N) and (b) precision-versus-recall on all items.

4 CONCLUSIONS

Quaternion-based recommendation algorithms are promising methods to overcome the sparsity problem of recommender systems. The proposed method, Q-SIMLP, relies on such a link prediction approach with the weights in the graph represented by quaternion numbers that precisely separates the "like" and "dislike" between a user and an item node, and distinguish "similarity" and "dissimilarity" between two users (or two items) nodes. The experimental results show that the Q-SIMLP method performs better than the remaining complex number-based algorithms, such as SIMLP and CORLP, regarding coverage and hits rate on the MovieLens and MovieLens Hetrec datasets. The obtained improvements of Q-SIMLP are attributed to the inclusion of similarity and dissimilarity factors between users and items, as well as like and dislike relationships between users and items. The Anova results indicate that the proposed Q-SIMLP algorithm is significantly better than CORLP and SIMLP methods in graph-based recommendation algorithms.

In addition, the Q-Hybrid recommendation method performs better than the proposed Hybrid-SIMLP algorithm in (Kurt, 2020), regarding hit-ratio, recall, and precision on the real-world Amazon sub-datasets. The improvements of our proposed method are attributed to the inclusion of similarity and dissimilarity factors between users' feature and items' feature vectors. The experimental results show that our approach demonstrates superior performance on real-world datasets compared to other algorithms. Furthermore, the proposed algorithm is adaptable by incorporating different information sources. In conclusion, Q-Hybrid can effectively deal with the deficiencies in other hybrid algorithms thanks to its improved design.

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Toolbox website: Quaternion toolbox for Matlab, http://qtfm.sourceforge.net/.