Revisiting Johann Bernoulli's Method for the Brachistochrone Problem

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Abstract: This paper reviews Johann Bernoulli's solution to the Brachistochrone problem, using an analogy to the movement of light and Fermat's principle of least time. Bernoulli's method is later used to derive solutions to some generalizations of the Brachistochrone problem. The problems solved using Bernoulli's method are the classical flat gravity Brachistochrone, spherical gravity outside the earth, and spherical gravity inside the earth ('gravity train').

1 INTRODUCTION

The Brachistochrone problem, meaning in Greek "shortest time, is the question regarding what is the shape of the path to slide a point mass between two arbitrary points with a height difference in the shortest time possible, while considering only the action of a constant gravitational force applied on it. Its formulation is considered as the birth of optimal control theory. Johann Bernoulli proposed to solve the problem using an analogy to light (de Icaza, 1993). According to Fermat's principle of least time, light will manage to find the optimal course in order to travel between two points at the shortest possible time. When the points lie in different mediums the light would refract and change its direction when passing between the mediums in order to maintain this principle. The relation between the light velocities in each medium, and the direction of the light movement is expressed through Snell's law. When used in spherical coordinates, Snell's law can also be generalized.

At a later date, the problem was solved again using a different approach, with variational calculus (Grasmair, 2010). This method's purpose is to find the optimal solution by minimizing the cost function of the traveling time, and by this to find the route which would provide the shortest time of travel between the points. Both Bernoulli's method, and calculus of variations provided the same solution. In this paper, several generalizations of the Brachistochrone are analysed with Bernoulli's method, and are validated using the calculus of variations method.

The first generalization considered is for a giant Brachistochrone outside earth, where the gravity varies with the radius, and the position relative to the earth's center. The problem was solved both by Bernoulli's method (Parnovsky,1998), and with calculus of variations (Mitchell, 2006). To this end, the derivation of Snell's law inside a sphere is provided (this derivation has not been found by the authors in the literature.) Additionally, the problem is solved inside the earth for a solution of a 'gravity train'. This problem has been solved in the past using Calculus of Variations (Vanderbei, 2013). To the best of our knowledge, a solution based on Bernoulli's method has not been published yet. This paper provides this solution and obtains an equivalent result.

Thus, the main contributions of this paper are threefold: i. A tutorial revisit of known solutions by the Bernoulli's method; ii. A detailed derivation of Snell's law in a sphere; iii. A new solution based on the Bernoulli's method.

2 CLASSIC BRACHISTOCHRONE

Bernoulli used an analogy between the motion of the point mass on the surface, and a motion of a light beam between infinitely many varying mediums (Fig. 1).

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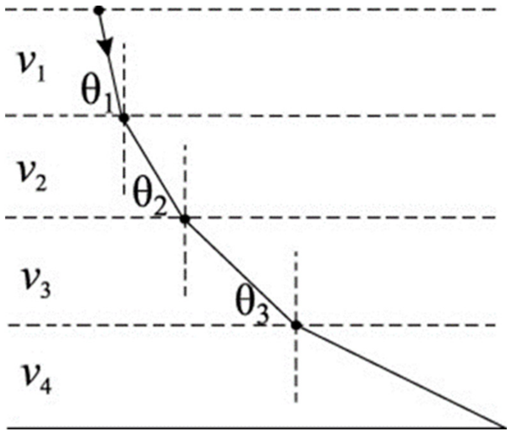

Figure 1: Light's movement through varying mediums.

Assume a point mass travels from point A to point B using only the gravitational force. Set a Cartesian coordinate system such that A is located at (0,0), and B at known (L,H) beneath point A. θ is the angle between the tangent to the surface and y axis (Fig. 2).

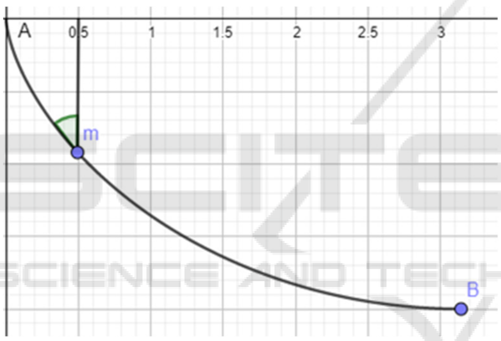

Figure 2: Mass course of movement.

Since the only force applied on the mass is gravity, the total energy is conserved.

$$E = V + T = -mgy + \frac{mv^2}{2} = const \tag{1}$$

$$E_A = 0 \tag{2}$$

$$-mgy + \frac{mv^2}{2} = 0 \tag{3}$$

$$\Rightarrow v = \sqrt{2gy} \tag{4}$$

From Snell's law (see Appendix B):

$$\frac{\sin\theta}{v} = const \tag{5}$$

$$\frac{\sin\theta}{\sqrt{2gy}} = const \tag{6}$$

Squaring both sides and adding g to the constant:

$$\frac{\sin^2\theta}{\mathrm{y}} = \mathrm{const} \tag{7}$$

This relation represents the differential equation of a cycloid. To show how, a geometric proof is provided (Levi, 2015). Consider the sketch in Fig. 3.

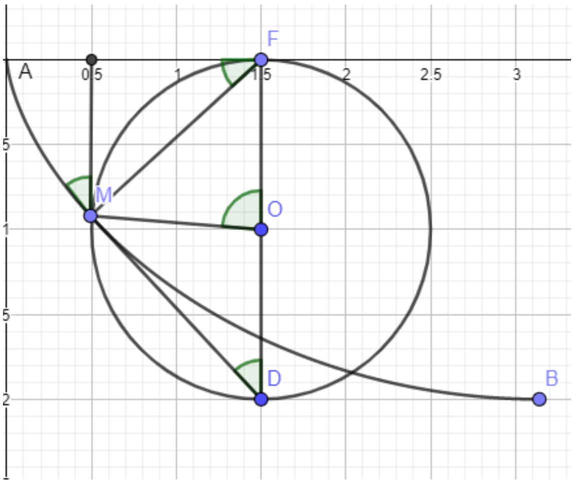

Figure 3: Geometric proof for Brachistochrone.

Since the cycloid is created from a rolling circle with a radius R, at any given time F is the instant center of rotation of the circle, so every point on the circle rotates around F at that moment and performs a circular motion around that point. M moves in a circular motion with respect to F at any given time, so it's velocity is perpendicular to the line MF. The velocity vector is in the same plane as the surface the mass slides on, so the tangent line of the surface at any given moment is perpendicular to MF. Continue the tangent on a straight line until it reaches the circle on point D, such that ∢FMD=90°. A circumferential angle that equals 90° lies on the diameter, so FD is the diameter of the circle. Define the angle ∢FDM=θ. θ is a circumferential angle, so the central angle that lies on the same arc, ∢FOM=2θ. θ is also the angle of the mass because they are parallel angles. The angle between a chord in the circle to the tangent of the circle is the same as the circumferential angle that lies on this chord from the other side, thus:

$$\sphericalangle\mathrm{FDM} = \sphericalangle\mathrm{MFA} = \theta \tag{8}$$

Using the law of sines:

$$\frac{MF}{\sin(\sphericalangle\mathrm{FDM})} = \frac{DF}{\sin(\sphericalangle\mathrm{FMD})} \tag{9}$$

$$\frac{MF}{\sin\theta} = \frac{2R}{\sin 90°} \tag{10}$$

$$\Rightarrow MF = 2R\sin\theta \tag{11}$$

$$\frac{y}{MF} = \sin(\sphericalangle\mathrm{MFA}) \tag{12}$$

$$\frac{y}{2R\sin\theta} = \sin\theta \tag{13}$$

$$\frac{\sin^2\theta}{y} = \frac{1}{2R} = const \tag{14}$$

Thus Eq. (7) was verified. This equation represents the cycloid equation (see Appendix A):

$$\begin{cases} x = R(2\theta - \sin 2\theta) \\ y = R(1 - \cos 2\theta) \end{cases} \tag{15}$$

Where θ is the angle between the tangent to the surface and y axis, and is half the angle of the circle's rotation. R is the radius of the circle:

$$R = \frac{H}{2} \tag{16}$$

3 SOLVING THE BRACHISTOCHRONE PROBLEM FOR EXTERNAL SPHERICAL EARTH

Assume a spherical earth with a gravitational field:

$$\underline{g} = -\frac{M_{\oplus}G}{r^2}\underline{\hat{r}} \tag{17}$$

The center of the earth in ECI coordinates is at $\underline{r_O} = [0\ 0\ 0]^T$. It is required to find the course from point $A(x_A, y_A, z_A)$ to point $B(x_A, y_A, z_A)$) which a point mass would travel at the shortest time while applied only a gravitational force directed to $\underline{r_O}$.

$$\mathrm{r_A} \geq \mathrm{r_B} \tag{18}$$

Since the earth is assumed to be a perfect sphere, and the gravity is assumed to be only dependent on $_$r, there exists a coordinate system where A, and B both lie on the same plane. So using polar coordinates:

$$A(r_A, \theta_A), \qquad B(r_B, \theta_B) \tag{19}$$

The path of shortest time must satisfy Snell's law in a sphere (Parnovsky,1998) - see Appendix C:

$$\frac{r\sin\phi}{v(r)} = const \tag{20}$$

ϕ is the angle between the tangent to the surface and the radius vector. It was seen from energy conservation that v satisfies:

$$v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_A}\right)} \tag{21}$$

Therefore:

$$\frac{\mathrm{r}\sin\phi}{\sqrt{2\mathrm{GM}\left(\frac{1}{\mathrm{r}} - \frac{1}{\mathrm{r_A}}\right)}} = \mathrm{const} \tag{22}$$

$$\Rightarrow \sqrt{\frac{\mathrm{r}^3\mathrm{r_A}}{\mathrm{r_A} - \mathrm{r}}}\sin\phi = \mathrm{const} \tag{23}$$

In order to find the optimal path, it is required to identify the relation between ϕ, and θ.

$$\sin\phi = \frac{rd\theta}{dl} \tag{24}$$

$$\sqrt{\frac{r^3 r_A}{r_A - r}}\frac{rd\theta}{dl} = const = \sqrt{c} \tag{25}$$

Squaring both sides yields:

$$\frac{r^3 r_A}{r_A - r} r^2 d\theta^2 = cdl^2 = c[dr^2 + r^2 d\theta^2] \tag{26}$$

$$\Rightarrow \frac{d\theta^2}{dr^2} = \theta'(r)^2 = \frac{c(r_A - r)}{r^5 r_A - cr^2(r_A - r)} \tag{27}$$

Thus

$$\theta(r) = \pm \int_{r=r_A}^{r_B} \sqrt{\frac{c(r_A - r)}{r^5 r_A - cr^2(r_A - r)}}\,dr \tag{28}$$

The initial and terminal conditions:

$$\theta(r_A) = \theta_A, \qquad \theta(r_B) = \theta_B \tag{29}$$

The expression obtained via Bernoulli's method (Parnovsky, 1998) is equivalent to the variational calculus solution (Mitchell, 2006). Fig. 4 presents representative trajectories using (28).

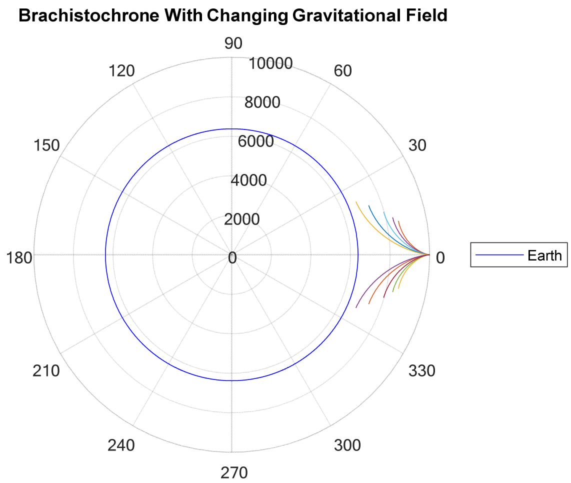

Figure 4: External spherical earth Brachistochrone.

4 SOLVING THE BRACHISTOCHRONE PROBLEM FOR INTERNAL SPHERICAL EARTH

Assume a spherical earth with an internal gravitational field (Levi, 2015):

$$\underline{g} = -\frac{GM}{R^3} r\underline{\hat{r}} \tag{30}$$

The gravitational potential is derived to be:

$$V(r) = \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right) \tag{31}$$

The kinetic energy of a point mass during its course:

$$E = T + V = \frac{1}{2}mv^2 + \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right) = const \tag{32}$$

At point A the mass starts the movement:

$$E_A = V_A = \frac{GMm}{2R}\left(\frac{{r_A}^2}{R^2} - 3\right) \tag{33}$$

$$\frac{1}{2}mv^2 + \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right) = \frac{GMm}{2R}\left(\frac{{r_A}^2}{R^2} - 3\right) \tag{34}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2R^3}({r_A}^2 - r^2) \tag{35}$$

$$\Rightarrow v = \sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)} \tag{36}$$

The path of shortest time must satisfy (13):

$$\frac{r \sin\phi}{v(r)} = const \tag{37}$$

ϕ is the angle between the tangent to the surface and the radius vector. It was seen from energy conservation that v satisfies:

$$v = \sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)} \tag{38}$$

Therefore:

$$\frac{r \sin\phi}{\sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)}} = const \tag{39}$$

$$\frac{r \sin\phi}{\sqrt{{r_A}^2 - r^2}} = const \tag{40}$$

The relation between ϕ and θ:

$$\sin\phi = \frac{r d\theta}{dl} \tag{41}$$

$$\frac{r\frac{r d\theta}{dl}}{\sqrt{{r_A}^2 - r^2}} = const \tag{42}$$

Squaring both sides yields:

$$\frac{r^4}{{r_A}^2 - r^2}\frac{d\theta^2}{dl^2} = c = const \tag{43}$$

$$\frac{r^4}{{r_A}^2 - r^2} d\theta^2 = c dl^2 = c(dr^2 + r^2 d\theta^2) \tag{44}$$

Thus

$$r'(\theta)^2 = \frac{dr^2}{d\theta^2} = \frac{r^4}{c({r_A}^2 - r^2)} - r^2 \tag{45}$$

The same expression has been obtained from both Calculus of Variations (Vanderbei, 2013), and from Bernoulli's method. Fig. 5 presents representative trajectories using (45).

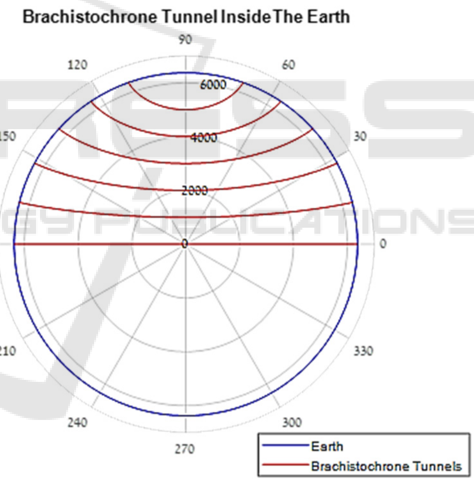

Figure 5: Internal spherical earth Brachistochrone 'Gravity Train'.

5 CONCLUSIONS

The paper presented and discussed the Brachistochrone problem, defined by Johan Bernoulli in 1696. The Brachistochrone problem was solved using Bernoulli's method of analogy to light. Additionally, the problem was generalized using several realistic influences and their effect on the Brachistochrone curve was derived and analysed. The Brachistochrone problem was solved for round earth solutions via Bernoulli's method. It has been shown that both methods, Calculus of Variations and Bernoulli's provide the same solution.

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APPENDICES

Appendix A - The path of a Cycloid:

In order to present the derivations of the Brachistochrone solutions it is first required to define the shape of the solution path, the Cycloid. Given a circle with radius R, rolling on a straight line on the x axis (Figs. 6-7). It is desired to form the equations of the path of a given point on the circle, initially located at $A(0,0)$.

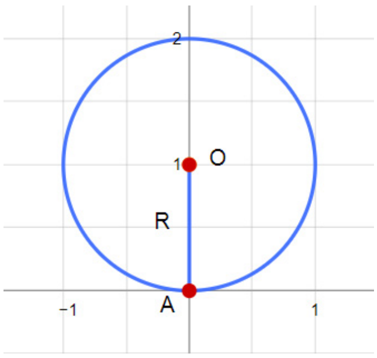

Figure 6: Point on circle before movement.

During the rolling of the circle, point A moves around the center O. Define θ as the angle between the segment OA, and the initial segment when $A(0,0)$. The length of OA is R as it is the radius of the circle. The center O position changes with respect to θ as the circle performs a pure roll by the following equations:

$$x_O = \theta R, \qquad y_O = R = const \tag{46}$$

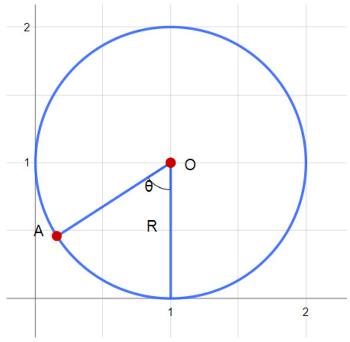

Figure 7: Point on circle after movement.

Using trigonometric relations, it is evident that:

$$x_A = x_O - Rsin(\theta) = \theta R - Rsin(\theta) \tag{47}$$

$$y_A = R - Rcos(\theta) \tag{48}$$

Therefore, the equations of a cycloid are:

$$\begin{cases} x = R(\theta - \sin\theta) \\ y = R(1 - \cos\theta) \end{cases} \tag{49}$$

Fig. 8 presents the shape of a Cycloid.

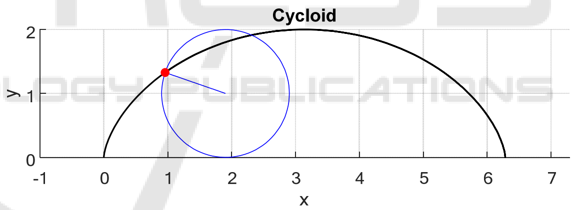

Figure 8: Shape of a cycloid.

Appendix B - Snell's law derivation (Fig. 9):

Snell's law states that when a beam of light travels between one medium to another it will deflect according to the following relation:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{50}$$

Where θ is the angle between the beam of light and the perpendicular to the medium transition line, and n is the refractive index - the ratio between the speed of light in vacuum and the speed of light in the given medium.

$$n = \frac{c}{v} \tag{51}$$

Snell's law is the implementation of Fermat's principle, which at the time was an empirical law claiming that light would find the path to travel between two given points at the minimal time. (At later dates, with a better understanding of the nature of light Fermat's principle was proven using Maxwell's equations of electromagnetism, and by the wave-particle duality using quantum mechanics.) Given points A, and B which lie in different mediums n_1, and n_2 accordingly. Define the horizontal length m between A, and B, and the vertical length l. The length between A and the medium transition line is a. Mark x as the horizontal length between A and the point of transition between the mediums, which is unknown.

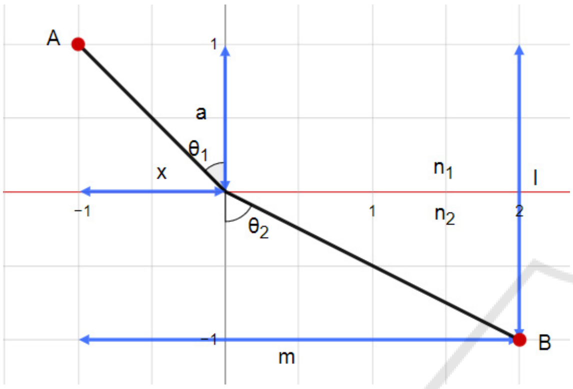

Figure 9: Refraction of light between mediums.

The velocity in the mediums n_1, and n_2, are v_1, and v_2 accordingly.

The distance between A, and the point of transition:

$$d_1 = \sqrt{x^2 + a^2} \tag{52}$$

The distance between B, and the point of transition:

$$d_2 = \sqrt{(l-a)^2 + (m-x)^2} \tag{53}$$

Since the light velocity is constant in each medium, the time of travel is:

$$t_1 = \frac{d_1}{v_1},\ t_2 = \frac{d_2}{v_2} \tag{54}$$

Thus, the total time of travel between A, and B is:

$$t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x^2+a^2}}{v_2} + \frac{\sqrt{(l-a)^2+(m-x)^2}}{v_2} \tag{55}$$

Using Fermat's principle, we wish to minimize the time as function of x:

$$\frac{dt}{dx} = 0 \tag{56}$$

$$\frac{dt}{dx} = \frac{2x}{2v_1\sqrt{x^2+a^2}} - \frac{2(m-x)}{2v_2\sqrt{(l-a)^2+(m-x)^2}} = 0 \tag{57}$$

$$\frac{x}{v_1 d_1} - \frac{(m-x)}{v_2 d_2} = 0 \tag{58}$$

$$\frac{x}{d_1} = \sin\theta_1\,, \qquad \frac{m-x}{d_2} = \sin\theta_2 \tag{59}$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{60}$$

Snell's law has been obtained:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{61}$$

By applying the relation $n = \frac{c}{v}$ the better-known equation is obtained:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{62}$$

<u>Appendix C</u> – Spherical Snell's law derivation:

Using Fermat's principle of minimum time, it is desired to compute Snell's law where the refractive index changes radially on an axis-symmetric sphere, so in polar coordinates:

$$n = f(r) \tag{63}$$

Given two mediums, one outside a sphere with radius $r = a$, and the second inside the sphere. The refractive index is therefore:

$$n(r) = \begin{cases} n_1, & r > a \\ n_2, & r < a \end{cases} \tag{64}$$

Choosing two arbitrary points: A is outside the sphere, and B is inside the sphere (Fig. 10).

$$r(\theta_A) = r_A, \qquad r(\theta_B) = r_B, \qquad \theta = \theta_B - \theta_A \tag{65}$$

It is required to find the point at which the light would choose to pass from n_1 to n_2 in order to travel from point A to point B at the minimum possible time.

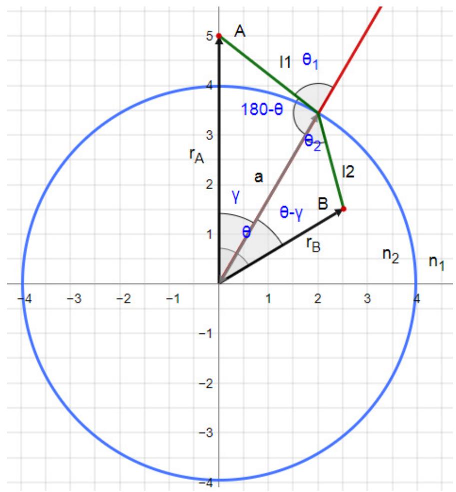

Figure 10: Refraction of light between spherical mediums.

The light would travel from point A to the sphere intersection point an angular distance of γ, the linear distance traveled over this angle is obtained via the cosine theorem:

$$l_1{}^2 = a^2 + r_A{}^2 - 2ar_A \cos\gamma \tag{66}$$

Similarly, for the distance from the intersection to B:

$$l_2{}^2 = a^2 + r_B{}^2 - 2ar_B \cos(\theta - \gamma) \tag{67}$$

The light travels in a given medium at a velocity of:

$$v = \frac{c}{n} \tag{68}$$

Where c is the speed of light in vacuum. The time that takes the light to cover the distances:

$$t_1 = \frac{l_1}{v_1} = \frac{\sqrt{a^2 + r_A{}^2 - 2ar_A \cos\gamma}}{v_1} \tag{69}$$

$$t_2 = \frac{l_2}{v_2} = \frac{\sqrt{a^2 + r_B{}^2 - 2ar_B \cos(\theta - \gamma)}}{v_2} \tag{70}$$

The total time of travel:

$$\begin{aligned} t &= t_1 + t_2 \\ &= \frac{\sqrt{a^2 + r_A{}^2 - 2ar_A \cos\gamma}}{v_1} \\ &+ \frac{\sqrt{a^2 + r_B{}^2 - 2ar_B \cos(\theta - \gamma)}}{v_2} \end{aligned} \tag{71}$$

Applying Fermat's principle of minimum time:

$$\frac{dt}{d\gamma} = 0 \tag{72}$$

$$\begin{aligned} &\frac{dt}{d\gamma} \\ &= \frac{ar_A \sin\gamma}{v_1\sqrt{a^2 + r_A{}^2 - 2ar_A \cos\gamma}} \\ &- \frac{ar_B \sin(\theta - \gamma)}{v_2\sqrt{a^2 + r_B{}^2 - 2ar_B \cos(\theta - \gamma)}} = 0 \end{aligned} \tag{73}$$

$$\frac{r_A \sin\gamma}{v_1 l_1} - \frac{r_B \sin(\theta - \gamma)}{v_2 l_2} = 0 \tag{74}$$

Using the sine theorem:

$$\frac{l_1}{\sin\gamma} = \frac{r_A}{\sin(\pi - \theta_1)} = \frac{r_A}{\sin\theta_1}, \quad \frac{l_2}{\sin(\theta - \gamma)} = \frac{r_B}{\sin\theta_2} \tag{75}$$

$$\frac{r_A \sin\theta_1}{v_1 r_A} - \frac{r_B \sin\theta_2}{v_2 r_B} = 0 \tag{76}$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{77}$$

$$\Longrightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{78}$$

This result is the same as the law derived in linear coordinates (82). Assume homogenous spherical medium (Fig. 11), the velocity inside the sphere is constant:

$$v(r) = const \tag{79}$$

Since there is no refraction were $n = const$ the fastest route would be a straight line, and that is the path in which the light travels.

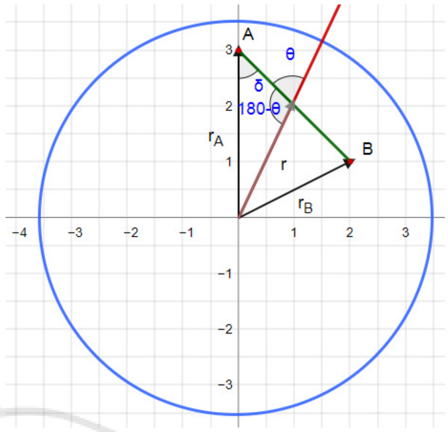

Figure 11: Movement of light inside spherical medium.

The angle between the trajectory of the light and the initial radius vector r_A is constant.

$$\delta = const \tag{80}$$

Using the sine theorem for every $r, \theta(r)$ throughout the course between points A, and B:

$$\frac{\sin\delta}{r} = \frac{\sin(180 - \theta)}{r_A} \tag{81}$$

$$r \sin\theta = r_A \sin\delta = const \tag{82}$$

Since $n(r)$ is constant as long as the movement is a straight line:

$$n(r) \cdot r \sin\theta = const \tag{83}$$

Assuming there is a medium change between points A, and B:

During the movement of the light through l_1, and through l_2 there is no medium change, so it has been showed that the equation holds for these parts of the course. At the point of the refraction it was proven that Snell's law applies:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{84}$$

Since the radius is the same on that point it can be multiplied on both sides of the equation:

$$n_1 a \cdot \sin\theta_1 = n_2 a \cdot \sin\theta_2 \tag{85}$$

Since the equation is true on the refraction points, and also between refractions, it applies throughout all of the movement between points A, and B. So, overall:

$$n(r) \cdot r \sin\theta = const \tag{86}$$