# On the Local Dominance Properties in Single Machine Scheduling Problems

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Abstract: We consider a non-preemptive single machine scheduling problem for a non-negative penalty function $f$. For this problem every job $j$ has a priority weight $w_j$ and a processing time $p_j$, and the goal is to find an order on the given jobs that minimizes $\sum w_j f(Cj)$, where $C_j$ is the completion time of job $j$. This paper explores the local dominance properties in this problem, which provide a powerful theoretical tool to better describe the structure of optimal solutions by identifying rules that at least one optimal solution must satisfy, reducing the search space from $n!$ to $n!/3^{n/3}$ schedules and providing insights to show the computational complexity status for problem with a convex penalty from a general framework, such as the problem of minimizing the sum of weighted mean squared deviation of the completion times with respect to a common due date and jobs with arbitrary weights.

## 1 INTRODUCTION

In this paper we consider a non-preemptive single-machine scheduling problem, which has several operations represented by a set $\mathcal{J}$ of $n$ jobs, where each job $j$ has a processing time $p_j \in \mathbb{N}$ and a priority weight $w_j \in \mathbb{Q}^+$ where the objective is to find a schedule $\sigma$ of a set $\mathcal{J}$ of jobs that minimizes $\sum_{j\in\mathcal{J}} w_j f(C_j)$, where $f$ is a given penalty non-negative function and $C_j$ is the completion time of each job $j \in \mathcal{J}$ with a value greater than or equal to $\sum_{\sigma_i \le \sigma_j} p_i$, where $\sigma_j$ is the order of job $j \in \mathcal{J}$ in the schedule $\sigma$.

We focus on the dominance properties, which provide a powerful theoretical tool to better describe the structure of optimal solutions by identifying rules that at least one optimal solution must satisfy. This information can be used to enhance different exhaustive algorithms to find an optimal solution by reducing the search space of $n!$ different schedules and pruning early ineffective partial solutions in several problems with convex, concave, and piecewise linear penalty function (Vásquez, 2015; Bansal et al., 2017), even when the penalty functions are non-monotone increasing (Pereira and Vásquez, 2017; Díaz-Núñez et al., 2018; Díaz-Núñez et al., 2019; Falq et al., 2021; Falq et al., 2022), highlighting the *left-shifted* property of any optimal schedule. This property means that all executions happen without idle time between times $t_0$ and $t_n$, with $0 \le t_0$ and $t_n = t_0 + \sum_{j\in\mathcal{J}} p_j$.

In this setting, we consider an instance $I$ containing two jobs $i, j \in \mathcal{J}$ and distinguish two kinds of properties.

- We say that jobs $i, j \in \mathcal{J}$ satisfy local precedence at time $t$ - denoted $i \prec_{\ell(t)} j$ - if whenever in a schedule $\sigma$ job $j$ starts at time $t$ and is followed immediately by job $i$ then the schedule $\sigma$ is not optimal.
- We say that jobs $i, j \in \mathcal{J}$ satisfy global precedence in the time interval $[a,b]$ - denoted by $i \prec_{g[a,b]} j$ — if whenever in a schedule $\sigma$ we have $a \le C_j - p_j \le C_i - p_i - p_j \le b$, then $\sigma$ is sub-optimal, no matter if $i, j$ are adjacent or not.

We use the notation $i \prec_g j$ as a shorthand for $i \prec_{g[a,\infty]} j$. In addition $i \prec_{\ell[a,b]} j$ means $i \prec_{\ell(t)} j$ for all $t \in [a,b]$.

For convenience, we denote $F(S)$ the objective value from the jobs of schedule $S$ and define the following function on the domain $t \in [0,\infty)$

$$\nu_{ij}(t) := \left(1 - \frac{w_j}{w_i}\right) f(t + p_i + p_j) + \frac{w_j}{w_i} f(t + p_j).$$

Let $S_1$ and $S_2$ schedules for $I$ of the form $S_1 = AijB$ and $S_2 = AjiB$, for some sets of jobs A and B. Let $t$ be the sum of processing time of all jobs in A. Thus,

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we have that $i \prec_{\ell(t)} j$ is equivalent to

$$\begin{aligned}0 <& F(S_2) - F(S_1)\\ =& w_j f(t+p_j) + w_i f(t+p_i+p_j)\\ &- w_i f(t+p_i) - w_j f(t+p_i+p_j)\\ =& w_i \left( \left(1 - \frac{w_j}{w_i}\right) f(t+p_i+p_j) \right)\\ &+ \frac{w_j}{w_i} f(t+p_j) - f(t+p_i)\\ =& w_i \left(\nu_{ij}(t) - f(t+p_i)\right),\end{aligned}$$

and then, the following equivalence hold

$$i \prec_{\ell(t)} j \equiv 0 < \nu_{ij}(t) - f(t+p_i).$$

## 1.1 Our contribution

We explore the local dominance properties in single machine scheduling problems. We show that the total number of solutions that satisfy the local dominance properties has an upper bound defined by $n!/3^{n/3}$, which is a dramatic improvement over the $n!$ different schedules of the search space. In addition, we study the NP-hardness for convex penalty in a general framework and address particularly the problem of minimizing the sum of weighted mean squared deviation of the completion times with respect to a common due date, whose computational complexity status is still open (Vásquez, 2014), providing some insights to show the computational complexity status based on dominance properties.

# 2 SEARCH SPACE AND LOCAL DOMINANCE PROPERTIES

Given an algorithm that uses some search tree procedure to solve the scheduling problem, we consider a node of the search tree specified by a partial schedule $S'$. Let $i, j$ be two jobs not in $S'$, and let $t$ be $t_n - \sum_{k \in S} p_k$. Then the descendants of this node include the partial right-to-left schedules $i + S'$ and $j + S'$. Now except in some degenerate cases (e.g. identical jobs $i$ and $j$) for comparable jobs exactly one of $i \prec_{\ell(t)} j$, $j \prec_{\ell(t)} i$ holds. This implies that exactly one of the sub-descendants partial right-to-left schedules $j + i + S'$ and $i + j + S'$ exists in the search tree. By considering the left-shifted property of any optimal schedule, note that here we used the fact that the partial schedule was extended from right to left, and the local precedence relations between $i$ and $j$ were done for the same time point, which would not have been the case, if the schedule were constructed from left to right.

The effect of this observation is that the number of nodes in the second level of the tree is upper bounded by $\binom{n}{2}$. Thus, by multiplying these numbers for all even levels, the total number of leaves of the tree is upper bounded by $n!/\sqrt{2^n}$. However, we improve this upper bound.

In order to prove the new upper bound, we introduce the following lemma.

**Lemma 1.** *Consider three jobs $i, j, k \in \mathcal{J}$ with $w_i > w_j$ and $i \prec_{\ell(t+p_k)} j$. The following expression holds*

$$\begin{aligned}\nu_{jk}(t+p_i) <& \\ &\frac{w_k}{w_i}\left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i w_k} f(t+p_k+p_i+p_j)\right)\\ &-\frac{w_k}{w_i}\left(\frac{w_j - w_i}{w_i} f(t+p_k+p_j) - f(t+p_k+p_i)\right).\end{aligned}$$

*Proof.* By case assumption $i \prec_{\ell(t+p_k)} j$, we have

$$\begin{aligned}0 <& \nu_{ij}(t+p_k) - f(t+p_k+p_i)\\ =& \left(1 - \frac{w_j}{w_i}\right) f(t+p_k+p_i+p_j)\\ &+ \frac{w_j}{w_i} f(t+p_k+p_j) - f(t+p_k+p_i)\\ =& \frac{w_j w_i}{w_k(w_i-w_j)}\left(\left(\frac{w_k(w_i-w_j)^2}{w_i^2 w_j}\right) f(t+p_k+p_i+p_j)\right)\\ &+ \frac{w_j w_i}{w_k(w_i-w_j)}\left(\frac{w_k(w_i-w_j)}{w_i^2} f(t+p_k+p_j)\right)\\ &- \frac{w_j w_i}{w_k(w_i-w_j)}\left(\frac{w_k(w_i-w_j)}{w_j w_i} f(t+p_k+p_i)\right).\end{aligned}$$

The last equality follows from the assumption $w_i > w_j$. Thus we have

$$\begin{aligned}0 <& \left(\frac{w_k(w_i-w_j)^2}{w_i^2 w_j}\right) f(t+p_k+p_i+p_j)\\ &+ \frac{w_k(w_i-w_j)}{w_i^2} f(t+p_k+p_j)\\ &- \frac{w_k(w_i-w_j)}{w_j w_i} f(t+p_k+p_i)\end{aligned} \quad (1)$$

We use the following equality

$$\begin{aligned}&\frac{w_k(w_i-w_j)^2}{w_i^2 w_j}\\ &= \left(\frac{w_k^2 w_i^2 - 2w_k^2 w_i w_j + w_k^2 w_j^2 + w_i^2 w_j w_k - w_i^2 w_j w_k}{w_i^2 w_j w_k}\right)\\ &= \left(\frac{w_k}{w_j} - \frac{2w_k w_i}{w_i^2} + \frac{w_k w_j}{w_i^2} + \frac{w_i^2}{w_i^2} - \frac{w_j}{w_j}\right)\\ &= \left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i^2} - \left(1 - \frac{w_k}{w_j}\right)\right),\end{aligned} \quad (2)$$

replacing it in expression (1). Then, we obtain

$$\begin{aligned}
0 <& \left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i^2}\right) f(t+p_k+p_i+p_j) \\
&- \left(1 - \frac{w_k}{w_j}\right) f(t+p_k+p_i+p_j) \\
&+ \frac{w_k(w_i - w_j)}{w_i^2} f(t+p_k+p_j) \\
&- \frac{w_k(w_i - w_j)}{w_j w_i} f(t+p_k+p_i)
\end{aligned}$$

$$\begin{aligned}
=& \left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i^2}\right) f(t+p_k+p_i+p_j) \\
&- \left(1 - \frac{w_k}{w_j}\right) f(t+p_k+p_i+p_j) \\
&+ \frac{w_k(w_i - w_j)}{w_i^2} f(t+p_k+p_j) - \frac{w_k}{w_j} f(t+p_k+p_i) \\
&+ \frac{w_k}{w_i} f(t+p_k+p_i) \qquad (3)
\end{aligned}$$

We reorder the terms of expression (3) and have

$$\begin{aligned}
&\left(1 - \frac{w_k}{w_j}\right) f(t+p_k+p_i+p_j) + \frac{w_k}{w_j} f(t+p_k+p_i) \\
=& \nu_{jk}(t+p_i) \\
<& \left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i^2}\right) f(t+p_k+p_i+p_j) \\
&+ \frac{w_k(w_i - w_j)}{w_i^2} f(t+p_k+p_j) + \frac{w_k}{w_i} f(t+p_k+p_i) \\
=& \frac{w_k}{w_i}\left(\frac{w_j w_k + w_i^2 - 2w_i w_k}{w_i w_k} f(t+p_k+p_i+p_j)\right) \\
&- \frac{w_k}{w_i}\left(\frac{w_j - w_i}{w_i} f(t+p_k+p_j) - f(t+p_k+p_i)\right),
\end{aligned}$$

concluding the proof. □

We now prove that the number of sequences that respect the local dominance property among three jobs is only two.

**Theorem 1.** *Fix the sequences with three comparable jobs $i$, $j$ and $k$ in some consecutive order, where $t_1$ and $t_1 - (p_i + p_j + p_k)$ are the completion time and starting time, respectively. The number of sequences that respect the local precedence among them is two.*

*Proof.* Without loss of generality, we adopt $t_1 = p_i + p_j + p_k$. We know that the total sequences induces by three comparable jobs is $3! = 6$. We denote $S_1, S_2, S_3, S_4, S_5$ and $S_6$ the sequences $ijk$,$ikj$,$jki$,$jik$,$kij$ and $kji$, respectively.

Note that the jobs $a$ and $b$ are comparable, and therefore one of $a \prec_{\ell(t)} b$, $b \prec_{\ell(t)} a$ holds.

Let $\rightarrow$ and $\rightrightarrows$ be the exclusion relations between two sequences with the same pair of jobs ordered in inverse form at the beginning and the end, respectively. These two exclusions capture the fact that a single order is possible between two comparable jobs at the same point in time.

By using the exclusion relations, we have:

$$S_1 \rightrightarrows S_2 \rightarrow S_5 \rightrightarrows S_6 \rightarrow S_3 \rightrightarrows S_4 \rightarrow S_1$$

Therefore, the sequences that respect the exclusion relations can be in two sets:

$$A = \{S_1, S_3, S_5\} \text{ or } B = \{S_2, S_4, S_6\}$$

In addition, we distinguish six priority weight orders among the jobs defined as follows:

$$\begin{aligned}
&o_1 : \{w_i > w_j > w_k\} \quad o_2 : \{w_k > w_i > w_j\} \\
&o_3 : \{w_k > w_j > w_i\} \quad o_4 : \{w_j > w_i > w_k\} \\
&o_5 : \{w_i > w_k > w_j\} \quad o_6 : \{w_j > w_k > w_i\}.
\end{aligned}$$

Now, we prove by contradiction that only 2 of 3 sequences of each set respect the local precedence among the jobs. First, we consider the set $A$ and $w_i > w_j$, which covers the order priority weights $o_1, o_2$ and $o_5$

Formally, suppose the sequence $S_1, S_3$ and $S_5$ with $w_i > w_j$ respect the local precedence among the jobs. We observe the job pairs with the same local precedence for two particular sequences (in bold font) and have:

1. $S_1 = i\mathbf{jk} \wedge S_5 = \mathbf{jk}i$ imply $j \prec_{\ell(t)} k$ for $t = p_i, 0$
2. $S_1 = \mathbf{ij}k \wedge S_3 = k\mathbf{ij}$ imply $i \prec_{\ell(t)} j$ for $t = p_k, 0$
3. $S_3 = \mathbf{ki}j \wedge S_5 = j\mathbf{ki}$ imply $k \prec_{\ell(t)} i$ for $t = p_j, 0$

In particular,

$$\begin{aligned}
f(p_i + p_j) <\nu_{jk}(p_i) :=& \left(1 - \frac{w_k}{w_j}\right) f(p_j + p_i + p_k) \\
&+ \frac{w_k}{w_j} f(p_i + p_k) \qquad (4)
\end{aligned}$$

$$\begin{aligned}
f(p_k + p_i) <\nu_{ij}(p_k) :=& \left(1 - \frac{w_j}{w_i}\right) f(p_j + p_i + p_k) \\
&+ \frac{w_j}{w_i} f(p_k + p_j) \qquad (5)
\end{aligned}$$

$$\begin{aligned}
f(p_j + p_k) <\nu_{ki}(p_j) :=& \left(1 - \frac{w_i}{w_k}\right) f(p_j + p_i + p_k) \\
&+ \frac{w_i}{w_k} f(p_j + p_i) \qquad (6)
\end{aligned}$$

We use two of three above inequalities. Without loss of generality, we consider Expressions (5) and (6). Thus, we have

$$
\begin{aligned}
& f(p_k+p_j)+f(p_i+p_k) \\
<& \nu_{ij}(p_k)+\nu_{ki}(p_j) \\
=& \left(1-\frac{w_j}{w_i}\right) f(p_i+p_j+p_k) \\
& +\left(1-\frac{w_i}{w_k}\right) f(p_i+p_j+p_k) \\
& +\frac{w_j}{w_i} f(p_k+p_j)+\frac{w_i}{w_k} f(p_j+p_i) \\
=& \left(2-\frac{w_j}{w_i}-\frac{w_i}{w_k}\right) f(p_i+p_j+p_k)+\frac{w_j}{w_i} f(p_k+p_j) \\
& +\frac{w_i}{w_k} f(p_j+p_i)
\end{aligned}
$$

$$
\begin{aligned}
=& \left(\frac{-w_j w_k-w_i^2+2 w_i w_k}{w_i w_k}\right) f(p_i+p_j+p_k) \\
& +\frac{w_j}{w_i} f(p_k+p_j)+\frac{w_i}{w_k} f(p_j+p_i) \qquad (7)
\end{aligned}
$$

We reorder the Expression (7) and have

$$
\begin{aligned}
& \frac{w_k}{w_i}\left(\left(\frac{w_j w_k+w_i^2-2 w_i w_k}{w_i w_k}\right) f(p_k+p_i+p_j)\right) \\
& -\frac{w_k}{w_i}\left(\frac{w_j-w_i}{w_i} f(t+p_k+p_j)-f(t+p_k+p_i)\right) \\
<& f(p_j+p_i),
\end{aligned}
$$

where the left-term is greater than $\nu_{jk}(p_i)$ by Lemma 1. This implies that $k \prec_{(p_i)} j$, which contradicts the case assumption given by Expression (4).

For the case $w_k > w_i$, which covers the order priority weights $o_2$, $o_3$ and $o_6$, the proof considers Expressions (4) and (6) in a similar way, which contradicts Expression (5). To end the analysis for the set A, we have $w_j > w_k$ covering order priority weights $o_1$, $o_4$ and $o_6$. Here, the proof is also by contradiction. Expressions (4) and (5), contradicting Expression (6).

Symmetrically, we prove the case when the set $B$ is considered. Suppose $S_2, S_4$ and $S_6$ with respect the local precedence among the jobs, we have:

1. $S_2 = i\mathbf{kj} \wedge S_6 = \mathbf{kj}i$ imply $k \prec_{\ell(t)} j$ for $t = p_i, 0$
2. $S_2 = \mathbf{ik}j \wedge S_4 = j\mathbf{ik}$ imply $i \prec_{\ell(t)} k$ for $t = p_j, 0$
3. $S_4 = \mathbf{ji}k \wedge S_6 = k\mathbf{ji}$ imply $j \prec_{\ell(t)} i$ for $t = p_k, 0$

In particular,

$$
\begin{aligned}
f(p_i+p_k) <\nu_{kj}(p_i) &:= \left(1-\frac{w_j}{w_k}\right) f(p_j+p_i+p_k) \\
&+\frac{w_j}{w_k} f(p_i+p_j) \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
f(p_j+p_i) <\nu_{ik}(p_j) &:= \left(1-\frac{w_k}{w_i}\right) f(p_j+p_i+p_k) \\
&+\frac{w_k}{w_i} f(p_j+p_k) \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
f(p_k+p_j) <\nu_{ji}(p_k) &:= \left(1-\frac{w_i}{w_j}\right) f(p_j+p_i+p_k) \\
&+\frac{w_i}{w_j} f(p_k+p_i) \qquad (10)
\end{aligned}
$$

Here, we analyse the different cases. For the case where $w_i > w_k$, we have Expressions (9) and (10) that contradicts Expression (8), covering the order priority weights $o_1$, $o_4$ and $o_5$. For the case $w_j > w_i$, we consider Expressions (10) and (8), which contradicts Expression (9). This covers the order priority weights $o_3$, $o_4$ and $o_6$. Finally, the order priority weights $o_2$, $o_3$ and $o_5$ are covered by the case $w_k > w_j$. This uses Expressions (8) and (9), contradicting Expression (10).

Therefore, only two of three sequences of set respect the local precedence among the jobs. Specifically, the pairs sequences are:

$$
\{S_1,S_3\} \quad \{S_1,S_5\} \quad \{S_3,S_5\} \quad \{S_2,S_4\} \quad \{S_2,S_6\} \\
\{S_4,S_6\} \quad \{S_1,S_6\} \quad \{S_2,S_3\} \quad \{S_4,S_5\},
$$

which concludes the proof. □

From Theorem 1, we obtain the following corollary.

**Corollary 1.** *Consider three jobs consecutively executed. The local precedence property is satisfied at most for two pairs of jobs at any time in* $[t_0, t_n]$.

Finally, Corollary 2 shows a new upper bound, which represents a dramatic improvement over the $n!$ different schedules of the search space.

**Corollary 2.** *Given an algorithm, which uses some search tree procedure to solve the scheduling problem. The total number of leaves of the tree is upper bounded by* $n!/3^{n/3}$

*Proof.* The proof follows the observation from Theorem 1, which implies that the number of nodes in the third level of the tree is upper bounded by $2\binom{n}{3}$ and for a multiplying argument, the total number of leaves of the tree is upper bounded by $n!/3^{n/3}$ □

# 3 PENALTY FUNCTIONS WHERE THE PROBLEM BECOMES EASY

In this section, we study the penalty functions where the problem becomes easy. We show that for any increasing convex penalty functions, an instance of $n$

jobs with equal Smith ratios, i.e. $w_i/p_i$ equal to a constant for all job $i = 1,\dots,n$, admits an optimal schedule where the jobs are ordered in non-increasing weight.

**Theorem 2.** *Consider a strict convex penalty function $f(t)$ and two jobs $i, j \in \mathcal{J}$. If $w_i/p_i = w_j/p_j$ and $p_i > p_j$, then $i \prec_\ell j$*

*Proof.* Let $A, B$ be two arbitrary job sequences. Suppose that $i, j \in \mathcal{J}$ are adjacent in an optimal schedule and let $t$ be the largest completion time of the jobs in $A$. The claim states that the order $i, j$ generates a cost strictly smaller than the order $j, i$, i.e.

$$
\begin{aligned}
&F(AjiB) > F(AijB)\\
\equiv &p_j f(t+p_j) + p_i f(t+p_i+p_j) > p_i f(t+p_i)\\
&+ p_j f(t+p_i+p_j)\\
\equiv &(p_i - p_j) f(t+p_i+p_j) + p_j f(t+p_j) > p_i f(t+p_i)\\
\equiv &(1-\frac{p_j}{p_i}) f(t+p_i+p_j) + \frac{p_j}{p_i} f(t+p_j) > f(t+p_i)\\
\equiv &\nu_{ij}(t) > f(t+p_i),
\end{aligned}
$$

which holds for $p_i > p_j$ and $f$ strictly convex. □

From Theorem 2, we obtain the following statement.

**Corollary 3.** *Consider a strictly convex penalty function $f(t)$ and jobs with equal Smith ratios. This problem admits an optimal schedule where the jobs are ordered in non-increasing weight.*

Note that Corollary 3 applies to the problem of minimizing the sum of weighted mean squared deviation of the completion times with respect to a common due date, where the penalty function is strict convex.

# 4 FUTURE RESEARCH

In literature, previous NP-hardness proofs for the scheduling problem with some convex and concave penalty function involved almost equal ratio instances (see (Jinjiang, 1992; Vásquez, 2014) for example). We now provide as future research some insights to show the computational complexity of the problem of minimizing the sum of weighted mean squared deviation of the completion times with respect to a common due date, whose status is open for jobs with arbitrary weights (Vásquez, 2014).

In practice, we define an instance $I^C$ as follows: Consider the strict convex penalty function $f(t) := (t-d)^2$ and a set $\mathcal{J}$ with $2n+1$ jobs where $\mathcal{B} \subseteq \mathcal{J}$ with $\mathcal{B} = \{1,\dots,2n\}$ with equal Smith ratios and $p_i > p_{i+1}$ for $i = 1,\dots,2n-1$ and a job $k = 2n+1$ with $w_k > p_k$, $p_k \geq \max_{i\in\mathcal{B}} p_i$, and $\sqrt{w_k \min_{i\in\mathcal{B}} p_i} \geq p_k$. Based on the global properties from Corollary 1 in (Pereira and Vásquez, 2017) and Theorem 1 in (Bansal et al., 2017), and Theorem 1, in the optimal solution of these instances $I^C$, the completion time of job $k$ belongs to the interval $(d - p_k - \min_{i\in\mathcal{B}} p_i, d + \min_{i\in\mathcal{B}} p_i)$, all jobs preceding $k$ are scheduled in a non-increasing processing time, and all jobs following $k$ are scheduled in a non-increasing processing time.

Based on Theorem 2, we focus on the necessary condition so that the problem does not become easy given by the sequence of three jobs that respects the local precedence among them, which is defined as follows: the job $k$ and two jobs $i'+1$ and $i'$, which are immediately executed before and after job $k$, respectively. Given the above sequence, we consider Theorem 1 and have that at most one of these job sequences defined by a) $k, i', i'+1$, b) $i', k, i'+1$ or c) $i', i'+1, k$, respects the local precedence among them. Clearly, the sequences a) and b) and, the sequences b) and c) are reciprocally excluded by local precedence between jobs $k$ and $i'$ and, jobs $k$ and $i'+1$, respectively. However, we note that we can exclude the sequence c) by choosing sequence a), and vice-versa. This exclusion is based on the definition of jobs $i', i'+1$, Lemma 3, the necessary condition and Corollary 1. This allows us reduce the search space, restricting the cases to be analyzed.

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