Non-linear Black-Scholes Option Pricing Model based on Quantum Dynamics

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Keywords: Non-linear Schrödinger Equation, Non-linear Black-Scholes Model, Option Pricing, Method of Lines, Model Calibration.

Abstract: This paper is concerned with the option pricing based on an extended non-linear Black-Scholes model. In literature, non-linear Black-Scholes models are formulated assuming the stochastic asset price volatility, volatile risk-free interest rate or the occurrence of the transaction costs. Since these assumptions are matching better the real market conditions, these models are regarded to be more accurate in option pricing than linear Black-Scholes model. In this paper the option pricing model is derived from non-linear Schrödinger equation. This equation governs the movement of quantum particles which is similar to the volatility of stock prices. The non-linear Schrödinger equation with external potential terms is formulated. The non-linear Black-Scholes option pricing model is formulated using the transformation of the non-linear Schrödinger equation from complex Hilbert to real Euclidean space. The developed model has been used to predict European call options price based on WIG20 stock prices. The model parameters have been estimated based on real market data. The method of lines has been used to solve numerically the non-linear option pricing model. The model parameters have been estimated based on real market data. Numerical results are provided and discussed. The obtained results confirm high accuracy of the proposed non-linear model.

1 INTRODUCTION

The financial market is complex and non-linear dynamic system. In order to reduce the risk of financial market operations the derivative instruments have been introduced. These instruments such as forward contracts, warrants or options are traded similarly as stocks, bonds or other securities. Among them the financial options are most frequently used as a tool minimizing the financial risk. The financial options are contracts that give the owner the right to buy or to sell an underlying asset at a fixed price and at a specific period of time. Buy and sell option contracts are called call and put contracts, respectively. The list of underlying assets includes stocks, bonds, commodities or any other financial instruments at a specified price within a specific time period. This period depending on the type of option contract could be as short as few days or as long as a couple of years. Without loss of generality, in this paper we shall confine to consider European options only. These type of option contracts can be executed only at a final time. Financial investors or accountants are mainly interested to determine the value and the price of the option contract.

In literature there are described different approaches to calculate price of the option contract. The first approach is based on the theory of martingale stochastic processes (Clark, 1973; Engle, 1982; Bollerslev, 1986; Hamilton, 1989; Hull et al., 1987). In this approach the starting point is the stochastic differential equation governing the volatility of the stock price. The discretized stochastic approach leads to binomial or trinomial (Rendelman et al., 1979) option pricing models where the stock price is assumed either to increase or to decrease with the same probability at a given discrete time instants. The next approach is based on the transformation of the stochastic differential equation into the deterministic partial differential equation (Peszek, 1995) governing the option contract price. This approach includes Black-Scholes option pricing model which is formulated in the form of linear parabolic boundary value problem.

Black-Scholes model describes time evolution of financial equity like European option (Black et al, 1973; Merton,1973; Ivancevic, 2011; Ivancevic,

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Wróblewski, M. and Myślinski, A.
Non-linear Black-Scholes Option Pricing Model based on Quantum Dynamics.
DOI: 10.5220/0011066000003197
In *Proceedings of the 7th International Conference on Complexity, Future Information Systems and Risk (COMPLEXIS 2022)*, pages 26-35
ISBN: 978-989-758-565-4; ISSN: 2184-5034

2010). It assumes that the underlying asset price S is a random variable depending on time t and following a geometric Brownian motion (Smoluchowski, 1906). Markets are assumed to be efficient, i.e., the market movements cannot be predicted (Fama, 1965). Moreover no arbitrage opportunity, i.e., no risk-free profit, as well as no transaction cost associated with the buying or the selling the option contract are assumed (Barles, 1998). Due to the application of Girsanov theorem as well as Itó formulas (Weinan et al., 2019) the stochastic differential equation governing the stock price volatility is transformed into deterministic linear parabolic equation, called Black-Scholes equation:

$$\frac{\partial V}{\partial t} = -\frac{1}{2}(\sigma S)^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV, \quad (1)$$

governing the market price evolution of the European stock option $V = V(t,S)$ in time interval $t \in [0,T], T > 0$ is given. The short term risk free interest rate $r > 0$ and the volatility rate σ are assumed to be real constants. The first and second order derivatives of option price V with respect to S are denoted by $\frac{\partial V}{\partial S}$, and $\frac{\partial^2 V}{\partial S^2}$, respectively. The parabolic equation (1) is completed by a suitable boundary condition. For the European call option this boundary condition takes the form:

$$V(T,S(T)) = \max\{0, S(T) - K\}, \quad S \in [S_0, S_T], \quad (2)$$

where $K > 0$ is a given strike price at final time $t = T$ and $S_0 = S(0)$, $S_T = S(T)$.

The assumptions of linear Black-Scholes option pricing model (1)-(2) are simplification of the real market conditions (Jankova, 2018). Especially the underlying stock price, its volatility, risk-free interest rate and the stock dividend are unknown, as well as they may rapidly change in short time interval with the high variance. This leads to high fluctuations in the calculated option prices with respect to their market valuation. These discrepancies cause that researchers and practitioners are looking for new option pricing models and/or methods. In literature are described many option pricing models based on the extension of the classical linear Black-Scholes model. Among others, there are many non-linear Black-Scholes models such as Heston models (Heston, 1993) where the stock volatility or the risk-free interest rate are assumed to be random rather than constant functions. The interesting class of option pricing models are fractional Black-Scholes models (El Hajaji, 2015), where the stock price is assumed to follow the geometric Lévy process.

The other way to obtain Black-Scholes pricing model is to use the quantum dynamics methods (Haven, 2004; Vukovic, 2015; Wróblewski, 2017). The similarity between quantum mechanics describing the micro world of atoms and particles (Einstein, 1924) and stochastic nature of stocks and the associated options, is based on observation, that quantum mechanics relies on realization of process generated by the probability function. In contrary, the stock price is driven by a stochastic process which is realization of the Brownian motion. These both processes have just one realization which indicates that there is a link between probability function and stochastic behavior of the stock price. Stock is always traded at certain prices what is interpreted as the realization of the stochastic process. The equivalence of the quantum and stochastic descriptions of the stock or option prices is shown in (Peña, 2020; Vukovic, 2015; Haven, 2004). Especially in (Vukovic, 2015) it is proved that Black-Scholes equation can be derived from Schrödinger equation by using tools of quantum mechanics. In (Haven, 2004) it has been shown that from the wave equivalent of the Black - Scholes option price model, a Black – Scholes type option price with a non-risk free return can be obtained using Black-Scholes methodology. This quantum approach has been also used in (Wróblewski, 2013) where the linear Black-Scholes equation has been transformed into linear Schrödinger equation. We have assumed that linear Black-Scholes model can be treated as equivalent to a free quantum particle in constant potential Hilbert space. This observation led to another approach in (Wróblewski, 2017), where the solutions to non-linear Schrödinger equation have been used for option pricing. Analytical solutions to the non-linear Schrödinger equation with different non-linear potentials, i.e., dark solitons, have been used to calibrate the model based on the market data. It allows to evaluate option price (Ivancevic, 2011; Ivancevic, 2010).

This paper is concerned with the development of new extended non-linear Black-Scholes option pricing model derived from the non-linear Schrödinger equation. This model will result from the transformation of non-linear Schrödinger equation from Hilbert into Euclidean space. The obtained Black-Scholes model will contain additional non-linear terms. Next, we shall apply method of lines (Sanchez, 2017) to find numerical solution. Finally we shall calibrate the obtained model using real market data. The paper is organized as follows. In section 2, we formulate a non-linear Schrödinger equation and discuss its features. The next section 3 describes the transformation of this equation into non-linear Black-Scholes equation. The method of lines and the results of its application are presented in section 4. Model calibration based on the market call options listed on the Warsaw Stock Exchange is provided in section 5. Finally remarks and conlusions as well as considered extensions of this paper are presented in section 6.

2 NON-LINEAR SCHRÖDINGER EQUATION

The Schrödinger equation is the fundamental model of the quantum dynamics (Towsend, 2012; Schneider et al., 2017). The non-linear form of this equation describes, among others, Bose-Einstein condensate (Bose, 1924) or non-liner optic phenomenon (Einstein, 1924; Einstein, 1925). The non-linear Schrödinger partial differential equation describing bosons is called Gross-Pitaevskii equation (Schneider et al., 2017; Roger-Salazar, 2013)and has the following form:

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\sigma\frac{\partial^2\psi(x,t)}{\partial x^2} + V_{ext}\psi(x,t) + \beta|\psi(x,t)|^2\psi(x,t), \quad (3)$$

where $x \in R$ and $t \in (0,T)$. The coupling constant β (Debnath, 2005) is proportional to the scattering length of two interacting bosons. In financial analysis it is interpreted as an adaptive market potential depending on the interest rate r. The imaginary number is denoted by $i = \sqrt{-1}$. Moreover $V_{ext} + \beta|\psi(x,t)|^2$ stands for the total potential energy with V_{ext} representing the external potential. We shall assume that the external potential $V_{ext} \neq 0$. For the existence results for non-linear Schrödinger boundary value problems see (Schneider et al., 2017; Hayashi, 2016).

Recall (Marquardt et al., 2020; Towsend, 2012), the wave function $\psi = \psi(x,t)$ defines a state of the quantum particle. Its absolute square $\mid \psi(x,t) \mid^2$ is the probability density function of finding particle in a given space and at a given time. Let us provide the quantum interpretation of the equation (3). Based on results in (Bose, 1924), it has been predicted in (Einstein, 1924) and (Einstein, 1925) that a phase transition in a gas of non interacting atoms could occur due to these quantum statistical effects. This phase transition period, Bose-Einstein condensation, would allow for a macroscopic number of non-interacting bosons to simultaneously occupy the same quantum state of lowest energy. Bose-Einstein condensate is described as wave function that has the form of product of wave functions of each particle. The stochastic behaviour of stock prices and bosons is similar.

In order to obtain new extended non-linear Black-Scholes equation from non-linear Schrödinger equation, we apply reverse transformation comparing to the one described in (Wróblewski, 2017). This approach is based on relations between the linear Schrödinger and the Black-Scholes partial differential equations (Wróblewski, 2013) as well as on the complex adaptive wave-form solution to the non-linear equation (Wróblewski, 2017). The imposed conditions on market efficiency or stock prices fluctuations are assume to hold. The extended model is expected to provide more accurate option valuation opportunities.

3 ENHANCED BLACK-SCHOLES MODEL

Using the non-linear Schrödinger equation (3) as well as the transformation of variables (x,t) into (S,t) we formulate the non-linear Black-Scholes model. The goal is to transform function $\psi(y,\tau)$ first into function $\psi(y,t)$, next into function $\psi(x,t)$ and finally into function $\psi(S,t)$.

Let us define first new variables, imaginary time τ

$$\tau \overset{def}{=} -it, \quad (4)$$

as well as

$$y \overset{def}{=} x - \left(r - \frac{\sigma^2}{2}\right)t, \text{ and } x \overset{def}{=} \ln(S). \quad (5)$$

Since the function ψ is interpreted as the option price having non-negative value we shall assume that $\psi(y,\tau) > 0$. Using (4) - (5) the non-linear Schrödinger equation (3) is written in new variables (y,τ) as:

$$i\frac{\partial\psi(y,\tau)}{\partial\tau} = -\frac{\sigma^2}{2}\frac{\partial^2\psi(y,\tau)}{\partial y^2} + V_{ext}\psi(y,\tau) + \beta\psi(y,\tau)^3. \quad (6)$$

We shall transform equation (6) from variables (y,τ) into original variables S and t. Using Wick rotation (4) in equation (6) we get:

$$\frac{\partial\psi(y,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2\psi(y,t)}{\partial y^2} + V_{ext}\psi(y,t) + \beta\psi(y,t)^3. \quad (7)$$

Now we will use transformation (5) to change variables in equation (7). Before we do it let us calculate additional derivatives of transformations from y to x and t. Consider first the transformation of $\psi(y,t)$ into $\psi(x,t)$. The first order derivative of ψ with respect to y is equal to:

$$\frac{\partial\psi(y,t)}{\partial y} = \frac{\partial\psi(x,t)}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial\psi(x,t)}{\partial t}\frac{\partial t}{\partial y}. \quad (8)$$

Second term in (8) is equal to zero because time variable t is independent variable and is not function of y, so we get:

$$\frac{\partial\psi(y,t)}{\partial y} = \frac{\partial\psi(x,t)}{\partial x}. \tag{9}$$

From (9) it follows, the second order derivative of function ψ with respect to y equals:

$$\frac{\partial^2\psi(y,t)}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\psi(y,t)}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\psi(x,t)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi(y,t)}{\partial y}\right) = \frac{\partial^2\psi(x,t)}{\partial x^2}. \tag{10}$$

The first order derivative of function ψ with respect to time t is equal to

$$\frac{\partial\psi(y,t)}{\partial t} = \frac{\partial\psi(y,t)}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\psi(y,t)}{\partial t} = \frac{\partial\psi(x,t)}{\partial x}\left(r - \frac{\sigma^2}{2}\right) + \frac{\partial\psi(x,t)}{\partial t}. \tag{11}$$

Inserting (10) and (11) in equation (7), we get:

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial\psi(x,t)}{\partial x}\left(\frac{\sigma^2}{2} - r\right) - \frac{\sigma^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + V_{ext}\psi(x,t) + \beta\psi(x,t)^3. \tag{12}$$

Using (5) in (12) we get:

$$\frac{\partial\psi(S,t)}{\partial S} = \frac{\partial\psi(x,t)}{\partial x}\frac{1}{S}. \tag{13}$$

It implies

$$\frac{\partial\psi(x,t)}{\partial x} = \frac{\partial\psi(S,t)}{\partial S}S. \tag{14}$$

The second derivative of function ψ in (12) with respect to x can be written in the equivalent form as the derivative with respect to S:

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi(S,t)}{\partial S}S\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi(S,t)}{\partial S}\right)S + 0 = \frac{\partial}{\partial S}\left(\frac{\partial\psi(x,t)}{\partial x}\right)S = \frac{\partial}{\partial S}\left(\frac{\partial\psi(S,t)}{\partial S}S\right)S = \left[\frac{\partial^2\psi(S,t)}{\partial S^2}S + \frac{\partial\psi(S,t)}{\partial S}\right]S = \frac{\partial^2\psi(S,t)}{\partial S^2}S^2 + \frac{\partial\psi(S,t)}{\partial S}S. \tag{15}$$

Inserting derivatives (14) and (15) into equation (12) we obtain

$$\frac{\partial\psi(S,t)}{\partial t} = \frac{\partial\psi(S,t)}{\partial S}S\left(\frac{\sigma^2}{2} - r\right) - \frac{\sigma^2}{2}\left[\frac{\partial^2\psi(S,t)}{\partial S^2}S^2 + \frac{\partial\psi(S,t)}{\partial S}S\right] + V_{ext}\psi(S,t) + \beta\psi(S,t)^3. \tag{16}$$

Next:

$$\frac{\partial\psi(S,t)}{\partial t} = S\frac{\sigma^2}{2}\frac{\partial\psi(S,t)}{\partial S} - rS\frac{\partial\psi(S,t)}{\partial S} - S^2\frac{\sigma^2}{2}\frac{\partial^2\psi(S,t)}{\partial S^2} - S\frac{\sigma^2}{2}\frac{\partial\psi(S,t)}{\partial S} + V_{ext}\psi(S,t) + \beta\psi(S,t)^3. \tag{17}$$

After reducing repeated parts in (17), and moving parts to the left, we are getting non-linear Black-Scholes equation:

$$\frac{\partial\psi(S,t)}{\partial t} + \frac{\partial\psi(S,t)}{\partial S}rS + \frac{\partial^2\psi(S,t)}{\partial S^2}\frac{(S\sigma)^2}{2} - V_{ext}\psi(S,t) - \beta\psi(S,t)^3 = 0. \tag{18}$$

Let us assume that the external potential is equal to the interest rate r (Wróblewski, 2013)

$$V_{ext} = r \tag{19}$$

From (18) and 19 we get:

$$\frac{\partial\psi(S,t)}{\partial t} + rS\frac{\partial\psi(S,t)}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2\psi(S,t)}{\partial S^2} - r\psi(S,t) - \beta\psi(S,t)^3 = 0. \tag{20}$$

with boundary conditions for European call option:

$$\psi(S,T) = \max(S(t=T) - K, 0). \tag{21}$$

Remark, setting in (20)-(21) $\beta = 0$, we obtain the linear Black-Scholes boundary value problem (1)-(2). The non-linearity of (20) follows directly from quantum dynamic equation (3).

4 NUMERICAL SOLUTION

The linear as well as non-linear Black-Scholes equations (1) and (20), respectively, have been solved numerically using the method of lines (Sanchez, 2017). The idea of this method is to replace the spatial derivatives in the partial differential equation (PDE) with finite difference approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. This leads to ordinary differential equation (ODE) system dependent on time variable only. Next this ODE will be solved using 4rd order Runge-Kutta method (Kutta, 1901).

4.1 Initial Condition

For the sake of simplicity in the performing of numerical calculations let us change final conditions (2) and

(21) for equations (1) and (20), respectively, to initial conditions. This can be achieved by introducing new time variable:

$$\zeta \stackrel{def}{=} T - t, \tag{22}$$

where $\zeta \in (T, 0)$. Using (22) the time derivative of function ψ is equal to:

$$\frac{\partial \psi(S,t)}{\partial t} = \frac{\partial \psi(S,\zeta)}{\partial \zeta}\frac{\partial \zeta}{\partial t} = -\frac{\partial \psi(S,\zeta)}{\partial \zeta}. \tag{23}$$

Substituting (23) into (20)-(21) and taking into account (22) we get non-linear Black-Scholes equation with initial rather than final condition:

$$\frac{\partial \psi(S,\zeta)}{\partial \zeta} = rS\frac{\partial \psi(S,\zeta)}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2 \psi(S,\zeta)}{\partial S^2} - r\psi(S,\zeta) - \beta\psi(S,\zeta)^3 = 0, \tag{24}$$

with the initial condition for European call option:

$$\psi(S, \zeta = 0) = \max(S(\zeta = 0) - K, 0). \tag{25}$$

Applying (22)-(23) to the linear Black-Scholes boundary value problem (1)-(2) we obtain:

$$\frac{\partial \psi(S,\zeta)}{\partial \zeta} = rS\frac{\partial \psi(S,\zeta)}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2 \psi(S,\zeta)}{\partial S^2} - r\psi(S,\zeta) = 0, \tag{26}$$

with the initial condition (25).

4.2 Finite Difference Approximation

Let us divide time interval $[0, T]$ and stock prices interval $[S_0, S_T]$ into N and M subintervals having the length δt and δS, respectively.

By $V[S_i, \tau_j]$ we denote the option price at S_i, $i = 0, 1, \ldots N$ and time τ_j, $j = 0, 1, ..M$. Therefore the discretized right hand side of linear Black-Scholes equation (25) has the form:

$$R_i = -rV[S_i, \tau_{j-1}] + rS(V[S_{i+1}, \tau_{j-1}] + V[S_i, \tau_{j-1}])/\delta S + 0.5\sigma^2 S^2 (V[S_{i-1}, \tau_{j-1}] - 2V[S_i, \tau_{j-1}] - V[S_{i+1}, \tau_{j-1}])/(\delta S)^2. \tag{27}$$

Taking into account the left part of equation (26) we obtain the discretized linear model:

$$\frac{\Delta V(S_i, \tau_j)}{\Delta \tau} = R_i. \tag{28}$$

The integration of equation (28) in time domain, provides its solution in time interval $\tau \in (0, T)$. Runge-Kutta coefficients for ODE equation (28) are as follows:

$$k_1 = \delta t R_i, \quad k_2 = \delta t(R_i + \frac{1}{2}k_1),$$

$$k_3 = \delta t(R_i + \frac{1}{2}k_2), \quad k_4 = \delta t(R_i + k_3). \tag{29}$$

Using (29) we can to calculate option value V for every price S_i and point time τ_j, according to formula:

$$V[S_i, \tau_j] = V[S_i, \tau_{j-1}] + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{30}$$

For non-linear Black-Scholes equation, the discretized right hand side of equation (24) is given in the form

$$R_i^* = -rV[S_i, \tau_{j-1}] - \beta V[S_i, \tau_{j-1}]^3 + rS(V[S_{i+1}, \tau_{j-1}] + V[S_i, \tau_{j-1}])/\delta S + 0.5\sigma^2 S^2 (V[S_{i-1}, \tau_{j-1}] - 2V[S_i, \tau_{j-1}] - V[S_{i+1}, \tau_{j-1}])/(\delta S)^2. \tag{31}$$

The procedure to solve numerically equation (24) is based on (29)-(30) where R_i is replaced by $R_i^\star$.

4.3 Numerical Results

The method of lines has been implemented in Python environment using numpy and matplotlib libraries (Kong, 2020) to solve numerically equations (1) and (20). The computations were performed using the following data: the range of stock prices is $S \in (0, 2000)$, the time interval is $\tau \in (0, 0.9)$ year, volatility $\sigma = 0.02$, interest rate $r = 0.01$, coefficient $\beta = 0.00001$, striking price $K = 1500$ PLN. The final condition has been set to $V(S, \tau = T) = \max(S - K, 0)$ and the imposed boundary conditions are as follows: $V(S = 0, \tau) = V(S = 2000, \tau) = 0$.

Numerical solutions for linear and non-linear Black-Scholes equations (1) as well as (20) are displayed on Figure 1 and 2, respectively.

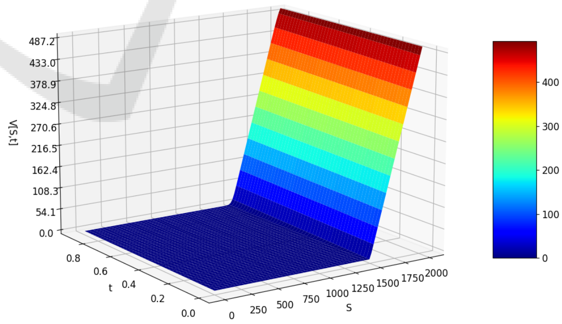

Figure 1: European call option price: linear model.

The solution to non-linear model displayed on Figure 2 differs from the solution to linear model on Figure 1. In the right part appears perturbation due to appearance of the non-linear term. We see that for non-linear model there is a visible bend near the strike price. It is more visible for time values closer to final time T. The solution obtained on Figure 1 matches the solutions reported in literature (Sanchez, 2017; Jankova, 2018).

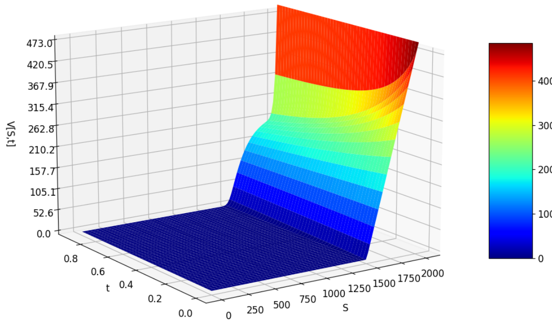

Figure 2: European call option price: non-linear model.

5 NON-LINEAR MODEL CALIBRATION

The accuracy of the non-linear option pricing model strongly depends on adaptive market potential parameter β. This parameter has been adjusted for market options listed on the Warsaw Stock Exchange where index WIG20 is the underlying asset. For the sake of model calibration the historical values of these options as well as their parameters have been acquired including values of $K, S_0, \sigma, r, \max(S_T - K, 0), T$. Parameter β in the non-linear Black-Scholes model (20) has been calibrated using bisection algorithm (Kong, 2020).

The aim of the calibration process is to find such value of parameter β to obtain the option price from non-linear model as close as possible to the option price acquired from the financial market. Parameter β is the unique unknown to determine. At the begining of computations the initial interval (min, max) is set where we expect to find the correct value of parameter β. So the first step is to divide this interval into two equal parts. The middle value from this interval is calculated and set as β. Option pricing calculations are performed. Next step is to check if the calculated and the market option prices are equal with a given tolerance. If so, then we can stop our calculation. If the calculated price is smaller than the desired one, then we change *max* value into the middle value, and again search parameter β in the range from *min* to the middle value. Otherwise, we change *min* value to the middle one, and keep searching between parameter β in the range from the middle to *max* values. This procedure is continued until the interval will shrink to a point. This algorithm requires $O(nLog(n))$ operations, where n denotes the number of iterations. The model (20) has been calibrated using market prices of options listed on the Warsaw Stock Exchange. These options and their parameters are displayed in Table 1.

The striking price K is given in PLN while the final time T in number of days in Table 1. Figures 3 - 9 display historical quotations of chosen options from Table 1 (left sub-figure) as well as of their underlying asset, i.e. index WIG20 of Warsaw Stock Exchange (right sub-figure). Remark, the graphs displaying the listed prices of the underlying asset on Figures 3 - 5, 6 - 7 and 8- 9 are nondecreasing, decreasing and non-monotonic functions, respectively. The quotation interval for options and assets on Figures 6 - 7 is very short. The chosen options are characterized by different length of time interval ranging from 7 till 70 days. The options characterized by diversified parameters have been specially chosen to allow for accurate model calibration.

Table 1: WIG20 based options.

Option	K	σ	r	T
OW20A211150	1150	0.065	0.010	43
OW20A211200	1200	0.045	0.010	70
OW20A211250	1250	0.230	0.010	22
OW20A211350	1350	0.035	0.010	7
OW20A211450	1450	0.002	0.010	9
OW20A212050	2050	0.020	0.010	67
OW20A212075	2075	0.024	0.010	25

The obtained computational results are summarized in Table 2. In this table the column ”Beta” provides values of calculated parameter β. Column V contains the market value of the option listed at a final time T and equal to difference between stock price S at the expiry date and the strike price K according to (21). Columns ”L” as well ”NL” provide values of options in PLN calculated based on linear model (1) and non-linear model (20), respectively. Columns denoted as ”(L-V)/V” as well as ”(NL-V)/V” provide relative errors in calculating the option prices with respect to the market option price V for linear and non-linear models, respectively.

Comparing the results provided by linear and non-linear models reported in Table 2 we can remark that for option OW20A211150 the expected market option price is 847.910 while predicted by linear and non-linear models 473.644 and 847.588, respectively. Similarly, for option OW20A211200 the expected price is 823.830, predicted by linear and non-linear models 456.909 and 823.195, respectively. For option OW20A211250 the market option price is 735.510 while calculated by linear and non-linear models 554.243 and 735.886, respectively. For options OW20A212050 and OW20A212075 the imposed striking price is higher than the final value of the underlying stock at $t = T$. It means that the value of the call option is equal to zero. The linear model provides the accurate option price evaluation equal to the market value. It seems that this model is sufficient

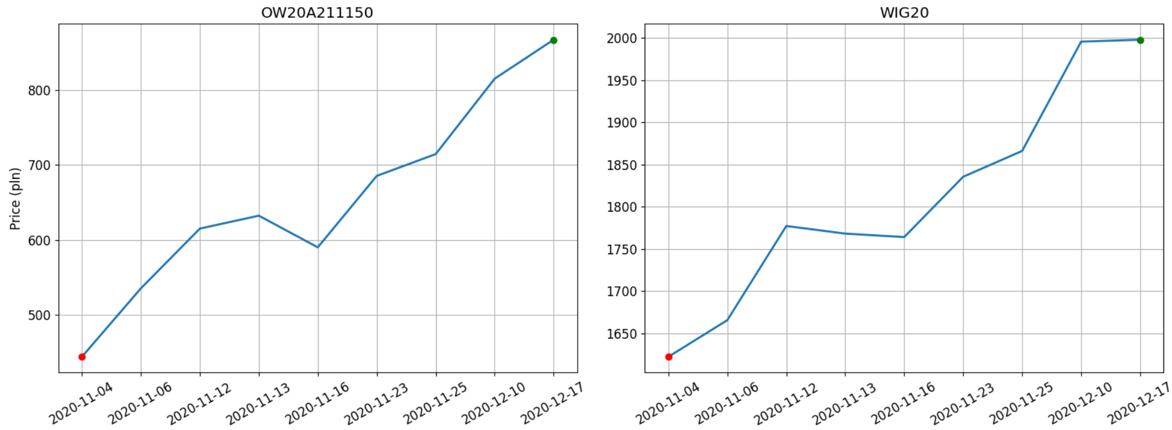

Figure 3: Historical data for OW20A211150 option (left) and WIG20 asset (right).

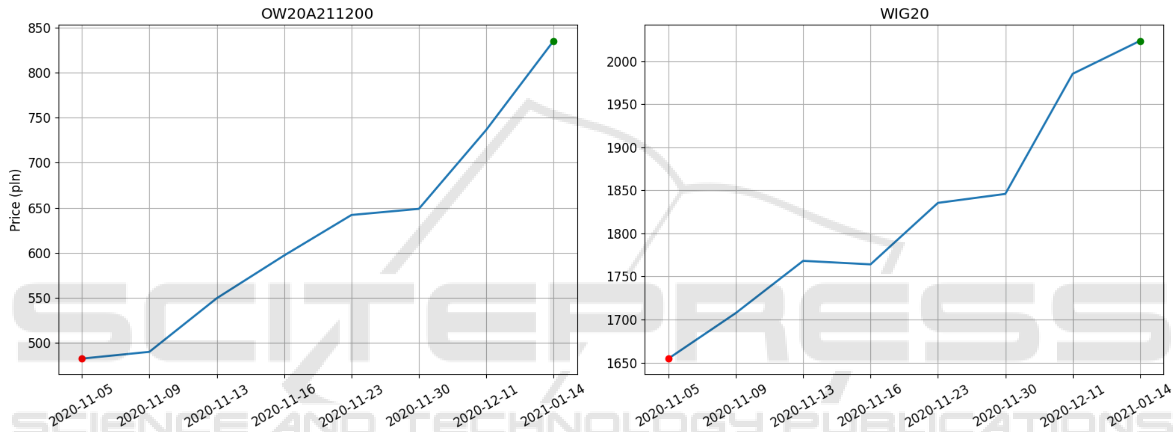

Figure 4: Historical data for OW20A211200 option (left) and WIG20 asset(right).

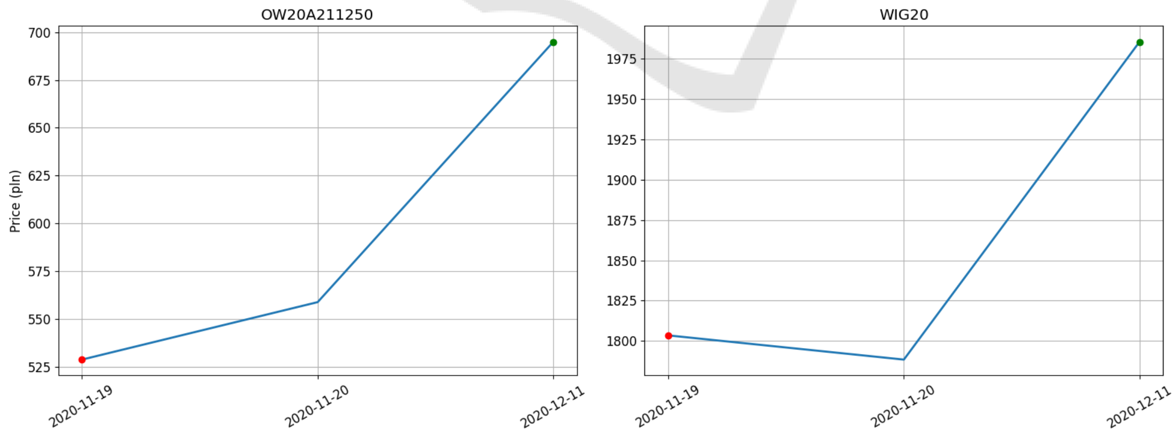

Figure 5: Historical data for OW20A211250 option (left) and WIG20 asset (right).

option valuation and it implies zero value of the parameter β. For theses options the non-linear model is equivalent to the linear one. The option prices obtained from the non-linear model were burdened with the relative error less than 0.5% with respect to the options market values. For options OW20A212050 and OW20A212075 the absolute error is equal to zero while the realtive error is not available.

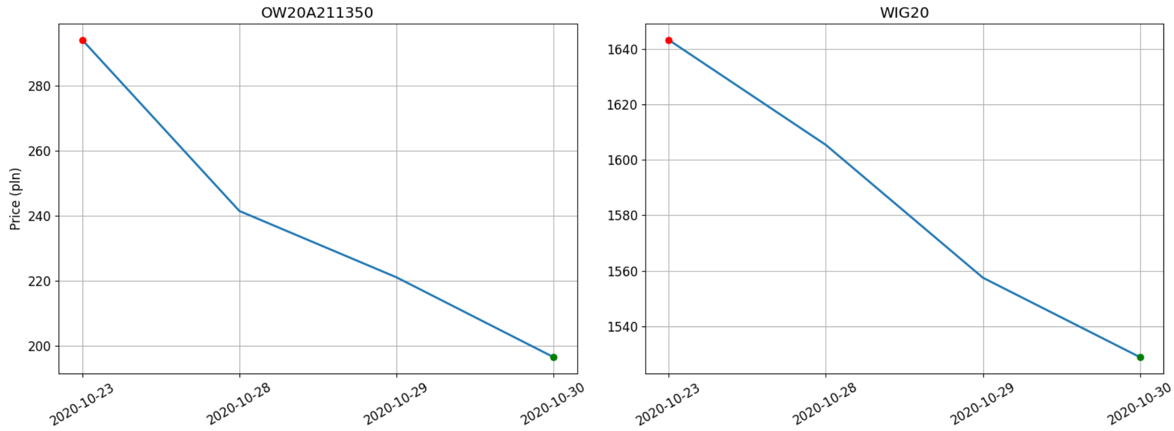

Figure 6: Historical data for OW20A211350 option (left) and WIG20 asset (right).

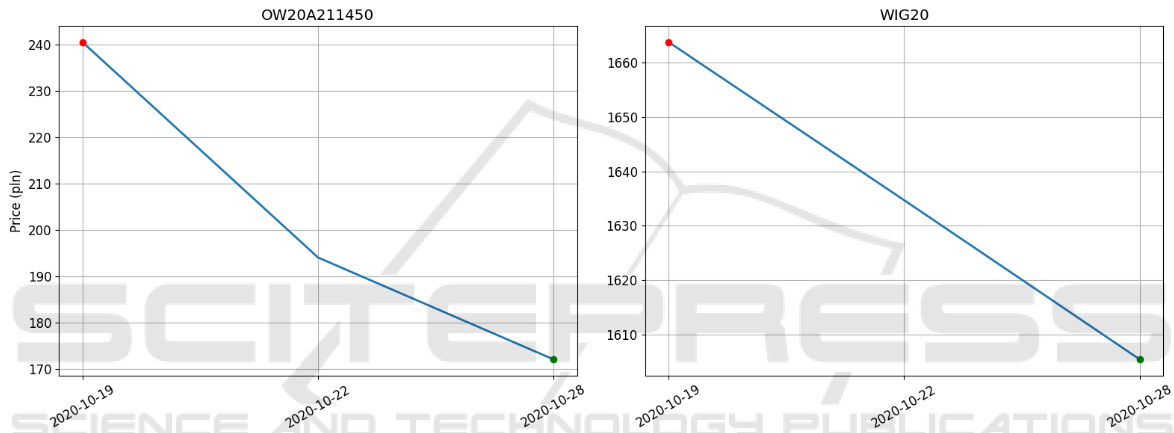

Figure 7: Historical data for OW20A211450 option (left) and WIG20 asset (right).

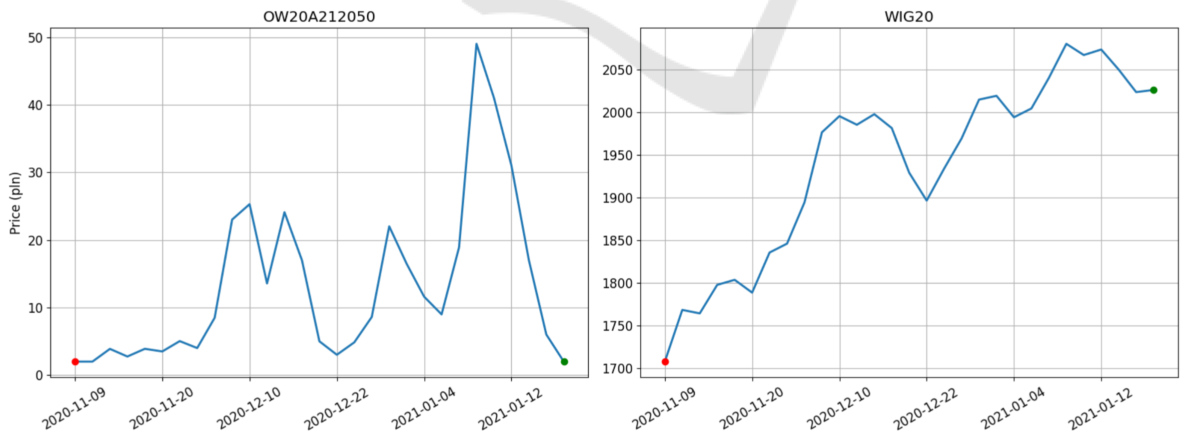

Figure 8: Historical data for OW20A212050 option (left) and WIG20 asset (right).

The proposed non-linear model provides much more accurate results than linear model with the volatility parameter σ taken from the market as asset price volatility. The linear model can provide more accurate results, but only under condition that parameter σ is calibrated. It is also interesting to observe that in general, parameter β takes low values. We have also calculated option price using linear Black-

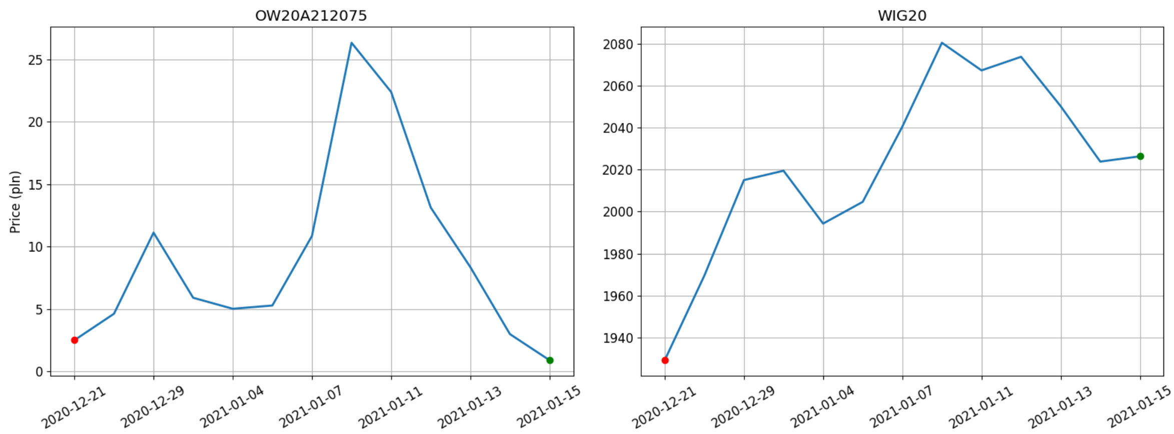

Figure 9: Historical data for OW20A212075 option (left) and WIG20 asset (right).

Table 2: Numerical results for parameter β calibration and option calculation.

Option	Beta	V=S-K	L	(L-V)/V	NL	(NL-V)/V
OW20A211150	-0.00000782	847.910	473.644	-44.139826 %	847.568	-0.040334 %
OW20A211200	-0.00000557	823.830	456.909	-44.538460 %	823.195	-0.077079 %
OW20A211250	-0.00000775	735.510	554.243	-24.645076 %	735.886	0.051121 %
OW20A211350	0.00008887	178.780	293.599	64.223627 %	178.721	-0.033001 %
OW20A211450	0.00011914	155.430	214.107	37.751399 %	155.522	0.059191 %
OW20A212050	0.00000000	0.000	0.000	n/a	0.000	n/a
OW20A212075	0.00000000	0.000	0.000	n/a	0.000	n/a

Scholes formula for the given set of parameters. In the case of option price predicted by this model is lower than expected one, i.e. is underestimated option price, then parameter $\beta < 0$. If the option price predicted by linear Black-Scholes model is higher than expected, i.e., overestimated option price, then parameter $\beta > 0$.

6 CONCLUSIONS

Quantum mechanics is describing the dynamics of micro world. It was an inspiration to create a new model describing option pricing using phenomenological approach. Extended non-linear Black-Scholes model for option pricing has been proposed.The model has been obtained by transforming into Euclidean space non-linear Schrödinger equation. The method of lines has been used to solve numerically classical linear and the proposed non-linear Black-Scholes models. Non-linear Black-Scholes model has been also calibrated based on market data, including historical volatility, risk-free rate, strike price and expiration time for a given options based on WIG20 index stock and listed on Warsaw Stock Exchange. Calibration was performed using divide and conquer algorithm to optimize the parameter β value.

The obtained results indicate that the proposed non-linear model for option pricing is much more accurate than the classical linear Black-Scholes model and provides option prices differing only less than 0.5% with respect to their market prices. In contrary, linear Black-Scholes model with volatility σ taken from the market provides very inaccurate results significantly different than market option prices. The computational complexity of the model equals $O(n \log n)$ and is relatively low. The developed model can be efficient calibrated in easy and efficient way. Since the parameter β is fairly well defined the model can be used for option forecasting. The proposed model is being extended to deal with American or Asian type options.

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