On the Helmholtz's Equation Model for Light Propagation in the Cornea

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Abstract: To model the incidence and reflection of light in the cornea we can use Maxwell's equations, which describe the electromagnetic wave's propagation field. In this paper we will focus on Maxwell's equations in the time harmonic form which translates in the Helmholtz's equation. We propose a numerical method based on nodal discontinuous Galerkin methods combined with a strategy which is specially designed to deal with curved domains which arise naturally in our domain of interest for the application.

1 INTRODUCTION

The cornea (see Figure 1) corresponds to the transparent part of the outer layer of the eye and its curved interface provides three-quarters of the eye's focusing power (the rest being provided by the lens). Thus, maintaining corneal curvature and transparency is essential for good vision, which is translated by less light reflection and therefore more information is captured.

The reasons that lead to corneal opacity are not yet completely determined, but there is consensus that corneal transparency is related to the shape, size and organisation of the of the stromal extracellular matrix and its elements, in particular collagen fibrils and their refractive indices, which translate the speed of light as it passes through the medium in question (see (Doutch et al., 2008), (Farrell et al., 1990), (Meek and Knupp, 2015) and the references therein). Maxwell's equations, which describe the electromagnetic field propagation, can be used to model incidence and reflection of light in the cornea. In this paper we will focus on Maxwell's equations in time harmonic form, and consequently, our formulation is based on Helmholtz's equation.

We will use a discontinuous Galerkin method (DG) to solve the Helmholtz equation (Hesthaven and Warburton, 2008). The DG method admits discontinuous solutions and it is a method with a high order of accuracy scheme. Moreover, being a local method it allows great flexibility when considering complex meshes. Analogously to the numerical flow in the finite volume method, which transports information from one local element to another, the numerical flow in the DG method connects adjacent elements, allowing to build the global approximation.

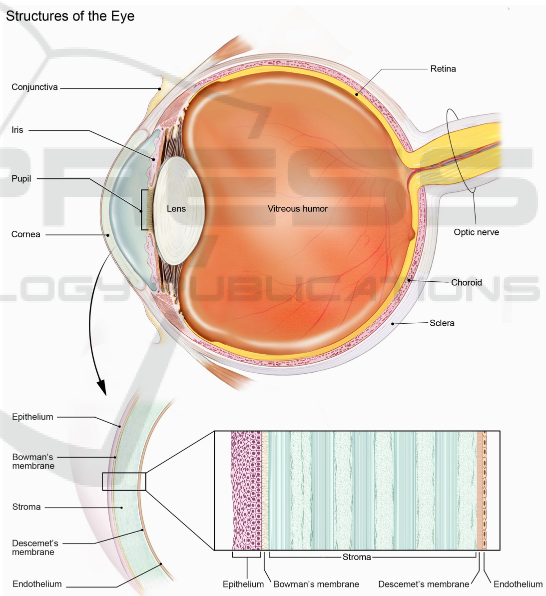

Figure 1: Anatomy of the human eye with corneal cross-section.
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Since we intend to solve the equation in a domain that mimics the cornea, polygonal meshes do not exactly fit the curved physical domain, thus reducing the the accuracy of the method. In order to overcome this

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problem, we will equip our numerical scheme with an optimisation method based on polynomial reconstruction (Costa et al., 2018).

This work is part of a more general problem that consists in understanding the physical basis for analysing the corneal transparency (Araújo et al., 2022).

2 ELECTROMAGNETIC WAVE EQUATION

Maxwell's equations are the fundamental set of equations that describe how the electromagnetic field propagates in free space and in any media. Assuming the electric and magnetic fields to be time harmonic,

$$\mathbf{E}(\mathbf{x},t) = e^{i\omega t}\hat{\mathbf{E}}(\mathbf{x}) \quad \text{and} \quad \mathbf{H}(\mathbf{x},t) = e^{i\omega t}\hat{\mathbf{H}}(\mathbf{x}),$$

Maxwell's equations can be written in the form,

$$i\omega\varepsilon_r\hat{\mathbf{E}} = \nabla\times\hat{\mathbf{H}}, \quad i\omega\mu_r\hat{\mathbf{H}} = -\nabla\times\hat{\mathbf{E}}, \tag{1}$$

$$\nabla\cdot\varepsilon_r\hat{\mathbf{E}} = 0, \quad \nabla\cdot\mu_r\hat{\mathbf{H}} = 0. \tag{2}$$

Equations (1) and (2) can be reduced to the Helmholtz's equation

$$-\nabla^2\hat{H}_z(x,y) = \omega^2\mu_r\varepsilon_r\hat{H}_z(x,y), \tag{3}$$

if we assume an homogeneous media and consider the TE-mode Maxwell's equations in 2D.

3 THE DG METHOD

To simplify the notation, let us consider equation (3) with a source term in the form

$$-\nabla^2 u(\mathbf{x}) - k^2 u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x}\in\Omega, \qquad u(\mathbf{x}) = 0,\ \mathbf{x}\in\partial\Omega, \tag{4}$$

where $\Omega\subset\mathbb{R}^2$ and $\mathbf{x} = (x,y)$. Let Ω_h be a conformal triangulation of Ω. Introducing a slack variable $\mathbf{q} = (q_x,q_y)^T$ such that $\mathbf{q} = \nabla u$, equation (4) can be written, in each triangle T_k of Ω_h, in the form

$$-\nabla\cdot\mathbf{q}(\mathbf{x}) - k^2u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{q}(\mathbf{x}) = \nabla u(\mathbf{x}).$$

We thus seek an approximation $(u_h^k,\mathbf{q}_h^k)$ to $(u,\mathbf{q})$ of the form

$$u_h^k(\mathbf{x}) = \sum_{i=1}^{N_p} u_h^k(\mathbf{x}_i^k)\ell_i^k(\mathbf{x})$$

and

$$q_{h,v}^k(\mathbf{x}) = \sum_{i=1}^{N_p} q_{h,v}^k(\mathbf{x}_i^k)\ell_i^k(\mathbf{x}),\ \ v = x,y.$$

where $\ell_i^k(\mathbf{x})$ is the Lagrangian polynomial of degree N defined on T^k by the grid points $\mathbf{x}_i^k$, $i = 1,...,N_p$, with $N_p = (N+1)(N+2)/2$.

Following (Hesthaven and Warburton, 2008), we conclude that the DG solution can be obtained by solving a system of linear equations

$$-\nabla S^k\mathbf{q}_h^k + \int_{\partial T^k}\hat{\mathbf{n}}\cdot\left(\mathbf{q}_h^k-\mathbf{q}_h^*\right)\ell^k\ d\mathbf{x} - k^2M^k\mathbf{u}_h^k = M^k\mathbf{f}_h,$$

$$M^k\mathbf{q}_{h,x}^k = S_x^k\mathbf{u}_h^k - \int_{\partial T^k}\hat{n}_x\left(\mathbf{u}_h^k-\mathbf{u}_h^*\right)\ell^k(\mathbf{x})\ d\mathbf{x},$$

$$M^k\mathbf{q}_{h,y}^k = S_y^k\mathbf{u}_h^k - \int_{\partial T^k}\hat{n}_y\left(\mathbf{u}_h^k-\mathbf{u}_h^*\right)\ell^k(\mathbf{x})\ d\mathbf{x},$$

where ∂T^k represents the boundary of T^k, $\mathbf{u}_h^k = [u_h^k(x_i^k,y_i^k)]_{i=1}^{N_p}$, $\mathbf{q}_{h,v}^k = [q_{h,v}^k(x_i^k,y_i^k)]_{i=1}^{N_p}$, $v = x,y$, $\ell^k(\mathbf{x}) = [\ell_i^k(\mathbf{x})]_{i=1}^{N_p}$, and, for $i,j = 1,\ldots,N_p$,

$$M_{ij}^k = \int_{T^k}\ell_i^k(\mathbf{x})\ell_j^k(\mathbf{x})\ d\mathbf{x}, \quad \nabla S^k = \left[S_x^k,S_y^k\right],$$

$$S_{v,ij}^k = \int_{T^k}\ell_i^k(\mathbf{x})\partial\ell_j^k(\mathbf{x})/\partial v\ d\mathbf{x}, \quad v = x,y.$$

The numerical fluxes are given by (Hesthaven and Warburton, 2008)

$$\mathbf{q}_h^* = \{\{\nabla u_h^k\}\}, \quad u_h^* = \{\{u_h^k\}\},$$

where the average

$$\{\{u\}\} = \frac{u^-+u^+}{2}.$$

We refer to the interior information of the element by a superscript "$-$" and to the exterior information by a superscript "$+$".

It is well know that the accuracy of the numerical method may be dramatically reduced when the boundary of the domain Ω is curved. In the next section we will consider an adaptation to the numerical scheme in order to preserve the optimal order of the DG method.

4 NUMERICAL SETTING

Since we are interested in simulating diverse scenarios which correspond to different kinds of organisation of the fibrils, polygonal domains don't fit on our regions of interest. In particular, we want to consider healthy and pathological scenarios that correspond to the organisation of the fibres represented in Figure 2.

A way to deal with curved boundaries is to consider the so-called isoparametric elements, introduced by Bassi and Rebay in the context of DG methods (Bassi and Rebay, 1997). The elements are called isoparametric, since the same functions are used to

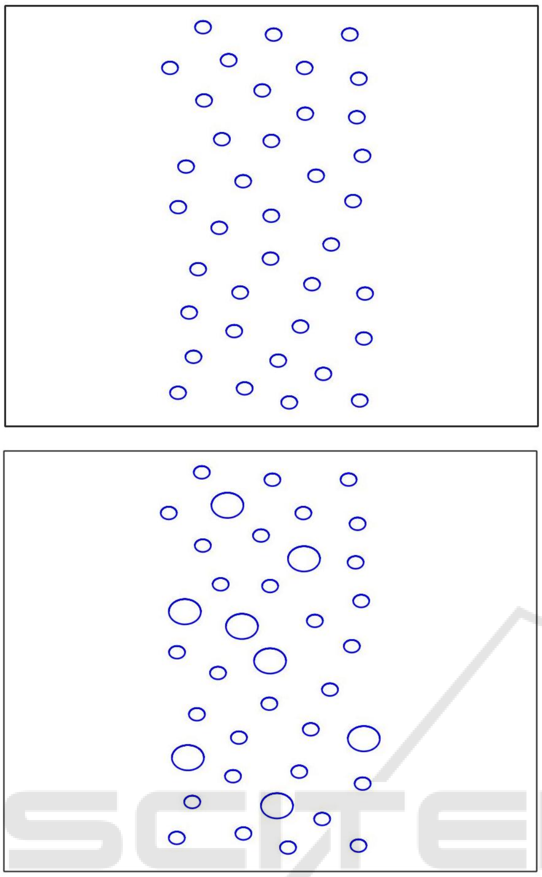

Figure 2: Stromal collagen fibres arrangement (Araújo et al., 2022). Top: healthy cornea; Bottom: cornea with some fibres whose diameter is doubled.

express the transformation from the reference element to the real element and the solution in the reference element. This approach requires the use of non-linear transformations of the reference triangle, which requires high computational effort. In addition, it requires the generation of curved meshes, which turns out to be impractical for complex geometries. In order to avoid nonlinear transformations, alternative approaches have been proposed, namely in the context of the finite volume method (see (Costa et al., 2018) and the references therein). In that work, a method called ROD (Reconstruction for Off-site Data) was considered. As the name of the method suggests, a polynomial reconstruction is developed which takes into account the real boundary conditions (which are not in the polygonal computational domain). This method does not require the generation of curved meshes to adjust the boundary, nor complex nonlinear transformations, which contributes for computational efficiency and simplifies the numerical schemes.

We now intend to generalize this approach to the DG method. The DG-ROD method starts with an iteration of the DG method, obtaining, for each $T^k \in \mathcal{T}_h$, the polynomial u_h^k. After this first iteration, for each T^k element with an edge e^k that intersects the boundary (see Figure 3), we determine a polynomial π^k that satisfies the boundary condition at a set of points on the boundary (Costa et al., 2018). Then we update the DG solution by imposing that on this edge e^k the boundary condition is given by the value of π^k. The procedure is repeated for a certain number of iterations.

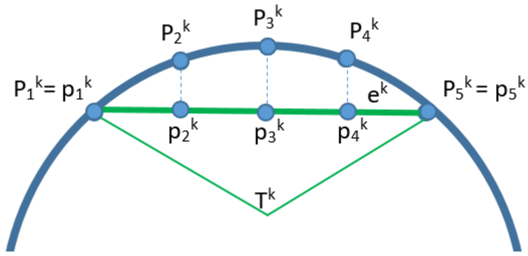

Figure 3: Example of an element T^k with an edge that intersects the boundary and a set of 5 points on the boundary.

The results obtained by the DG-ROD method are very encouraging and prove its effectiveness. We start by considering a particular example of (4) with $k = 1$, defined on a circular domain, with known exact solution u and apply both DG and DG-ROD. In particular, we are considering a simplified problem where the medium can be described by a circle of radius 1 centred at the origin and no internal fibrils.

If we denote by u_h an approximate solution determined with the numerical method, we say that the method has order of convergence p if

$$||u - u_h|| \leq Ch^p,$$

for a given norm $\|\cdot\|$, with C a real constant independent of h.

In order to estimate the order of convergence of both methods, we considered different spatial meshes generated by *Gmsh* (Geuzaine and Remacle, 2020), with different mesh parameters h. Considering two distinct values of h, say h_1 and h_2, we compute the maximum norms

$$E_{\infty,1} = ||u - u_{h_1}||_\infty \quad \text{and} \quad E_{\infty,2} = ||u - u_{h_2}||_\infty.$$

Assuming

$$E_{\infty,1}/E_{\infty,2} = (h_1/h_2)^p,$$

we have that the order of convergence can be estimated by

$$p = \frac{\log(E_{\infty,1}/E_{\infty,2})}{\log(h_1/h_2)} \Rightarrow p \approx 2\frac{\log(E_{\infty,1}/E_{\infty,2})}{\log(K_2/K_1)}$$

with K_i the number of triangles of the mesh i, for $i = 1,2$, where $h_1/h_2 \approx (K_2/K_1)^{1/2}$.

The results obtained in our simulations by the DG method and by the iterative DG-ROD method for polynomials of degree N, with $N = 1,2,3,4$, show that the order of convergence for the DG method is $p = 2$ and for the DG-ROD method the order is $p = N$. We then proved that the DG-ROD method, unlike the classical DG method, allows to obtain high order in domains with curved boundary.

5 CONCLUSIONS

In this paper we discuss an approach for dealing with the decreased accuracy of discontinuous Galerkin finite element method (DG) in domains with curved boundary. Following (Costa et al., 2018), we consider an approach based on the polynomial reconstruction of the boundary condition imposed on the computational domain, where the associated parameters are determined such that the reconstructions adequately satisfies the boundary condition imposed in the real domain. The overall method consists on an iterative method that considers two independent steps: solving the differential equation by the classical DG and the reconstruction process on the triangles with vertex on the boundary of the real curved domain. The numerical results obtained show the efficiency of the method, i.e., it is able to achieve high order of precision in domains with curved boundaries.

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